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NONLINEAR FILTERING AND APPROXIMATION TECHNIQUES

Dr. E. Pardoux

Final Report

— *October 1988* —

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This research concerned the theory of nonlinear filtering and numerical approximation in nonlinear filtering. The following results were obtained: 1) Under very general conditions, it is shown that the conditional density in nonlinear filtering is the unique solution, within an appropriate class of functions, of the Zakai equation. The main condition is that all coefficients are bounded and smooth. These coefficients are allowed to depend on the history of the observed process, 2) Developed a Lie algebraic criterion for the non-existence of finite dimensional filters, 3) Studied numerical methods for the approximate solution of Zakai's stochastic partial differential equations, 4) Developed approximate finite dimensional filters for high signal to noise ratio problems, 5) Compared two algorithms for maximizing the likelihood function associated with parameter estimation in partially observed diffusion processes. One of the algorithms is an EM algorithm and the other is based on the usual approach of computing and maximizing the likelihood function itself. This task yields to a comparison of nonlinear smoothing and CONT/....

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CONTENTS

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Contract DAJA 45-87-M-0296

FINAL REPORT

A) RESEARCH

The following results have been obtained by the contractors. Since all of them are concerned with the theory of - or numerical approximations in - nonlinear filtering, we first recall briefly what the problem of nonlinear filtering is.

Let $\{X_t, Y_t; t \geq 0\}$ be a pair of stochastic processes (for the sake of simplicity of the exposition, all the processes below will be one-dimensional) satisfying :

$$\begin{aligned} X_t &= X_0 + \int_0^t f(X_s)\, ds + \int_0^t g(X_s)\, dW_s \\ Y_t &= \int_0^t h(X_s)\, ds + \sigma V_t \end{aligned} \tag{0.1}$$

where $\{W_t, V_t; t \geq 0\}$ are two standard Wiener processes, which we shall mostly suppose to be independent, and X_0 is a random variable independent of $\{W_t, V_t; t \geq 0\}$. The process $\{X_t\}$ is unobserved. We observe $\{Y_t\}$, and seek to estimate X_t, given the information available at time t, i.e. given $\mathcal{Y}_t = \sigma\{Y_s; 0 \leq s \leq t\}$.

Note that the choice of the above model means that we have choosen $\{X_t\}$ to be a continuous Markov process, and that we observe :

$$y_t = h(X_t) + \sigma \xi_t$$

where $\{\xi_t\}$, the observation noise, is a white noise. The assumption that the observation noise is white (i.e. "δ-correlated") is crucial : it is the only case which can be solved (besides those which can be reduced to that one via appropriate transformations). The above model is obtained by applying the transformation :

$$Y_t = \int_0^t y_s\, ds$$

The reason for this is to avoid the handling of generalized processes (white noise is not a process in the ordinary sense). However, in the case where the observation noise $\{\xi_t\}$ and the signal $\{h(X_t)\}$ are independent, the theory of nonlinear filtering has been recently developped in the white noise setting, i.e. using $\{y_t\}$ (and not $\{Y_t\}$) as the observed process, see Kallianpur-Karandikar [7].

Going back to our model, the "best" estimate of any function of the unknown r.v. X_t say $\phi(X_t)$ based on $\mathcal{Y}_t$ is the conditional mean :

$$E[\phi(X_t)/\mathcal{Y}_t]$$

and computing that quantity for "any" function reduces to computing the conditional law of X_t given $\mathcal{Y}_t$. Assuming that this conditional law has a density $q(t,x)$, it is well known (see e.g. Pardoux [13]) that $q(t,x) = \left(\int_R p(t,x)\,dx\right)^{-1} p(t,x)$ where the "unnormalized conditional density" $p(t,x)$ solves the following stochastic partial differential equation, called Zakai's equation :

$$\begin{aligned} d_t p(t,x) &= L^* p(t,x)\,dt + h(x)p(t,x)\,dY_t,\ t \geq 0 \\ p(0,x) &= p_0(x) \end{aligned} \tag{0.2}$$

where $p_0(x)$ is the density of the law of X_0, and L^*, the "backward generator" of X_t, is the adgoint of the "forward generator":

$$L = \frac{1}{2} g^2(x) \frac{d^2}{dx^2} + f(x) \frac{d}{dx}$$

Note that at each time t, $p(t,\cdot)$ is a (random) function of x, i.e. a (random) element of an infinite dimensional space. This is of course a serious problem for practical algorithms.

Let us now describe the results which we have obtained during the period covered by the present contract.

1) A uniqueness theorem for Zakai's equation

It is of interest to give conditions under which the "unnormalized conditional density" is the unique solution of equation (0.2), within a certain class of processes. Such a uniqueness result is obtained by M. Chaleyat-Maurel, D. Michel, E. Pardoux [3], under the condition that the coefficients f, g and h be bounded and smooth (they are allowed at each time t to depend on the past of the observed process $\{Y_s\}$, which is very important for applications in stochastic control); the two Wiener processes $\{W_t\}$ and $\{V_t\}$ in (0.1) are allowed to be correlated.

The uniqueness was known until now only under additional restrictions, either the non degeneracy of L, or the independence of $\{W_t\}$ and $\{V_t\}$. Note that in the latter case, uniqueness is known to hold even with an unbounded observation function h.

2. Non existence of a finite dimensional optimal filter

We noted above that the solution of Zakai's equation takes values in an infinite dimensional space. It could in fact happen that the solution varies only in a finite dimensional subspace of that infinite dimensional space.

This is indeed the case in all the situations where a finite dimensional optimal (or "exact") filter is known to exist, i.e. in the linear case and in a few extensions of the linear case, see Haussmann-Pardoux [6].

It has been conjectured at the beginning of this decade, and then rigorously proved that under some mild conditions which are satisfied in most nonlinear situations, there is no finite dimensional equation driven by the observation such that the conditional law would be a function of its solution.

This means essentially that the solution of Zakai's equation does not stay in any finite dimensional space, or that its probability law fills in the function space in which it lives. Such a property is very close to the kind of properties which can be proved for (finite dimensional) stochastic differential equations via the Malliavin calculus.

Ocone [4] has developped a Malliavin calculus analysis of stochastic partial differential equations and applied it to nonlinear filtering. Ocone and Pardoux [5] improve the application to nonlinear filtering, in that the criterion is much easier to check on practical examples.

3. Time discretization of Zakai's equation

The research reported in the previous section shows that, in most cases, there is no chance that the solution of Zakai's equation can be solved by means of a finite number of statistics built from the observation process. Therefore, it is worth studying numerical methods for the approximate solution of Zakai's equation, i.e. the stochastic partial differential equation:

$$\begin{aligned} dp_t &= L^* p_t \, dt + h p_t \, dY_t \\ p_0 &= \bar{p} \end{aligned}$$

Moreover, this could provide a reference method with which to compare some approximate nonlinear filters such as those described in the next section.

Le Gland [9] has studied the problem of time-discretization. The idea is to use some kind of Trotter product formula and then to study the rate of convergence with respect to the time step Δt.

Let $0 = t_0 < \cdots < t_i < \cdots < t_n = T$ be a uniform partition of $[0,T]$ with time step Δt.

A first scheme is the following :

$$\bar{p}_{i+1} = \exp(h\Delta Y_i - \frac{\Delta t}{2} h^2) \; F^*_{\Delta t} \bar{p}_i$$

where $\Delta Y_i = Y_{t_{i+1}} - Y_{t_i}$ is the way the observation process is used, and $P^*_{\Delta t}$ is the semi-group with generator L^*. It is proved that :

$$\left\{ E \int |\bar{p}_i(x) - p(t_i, x)|^2 \, dx \right\}^{1/2} \leq C \Delta t$$

The second scheme is:

$$\tilde{p}_{i+1} = \exp(h\xi_i^\sharp - \frac{\Delta t}{4}h^2)\Phi_{\Delta t}(P_{\Delta t}^*)\exp(h\xi_i^\flat - \frac{\Delta t}{4}h^2)\tilde{p}_i$$

where $\xi_i^\sharp = \frac{1}{\Delta t}\int_{t_i}^{t_{i+1}}(s-t_i)\,dY_s$ and $\xi_i^\flat = \frac{1}{\Delta t}\int_{t_i}^{t_{i+1}}(t_{i+1}-s)\,dY_s$ are the way the observation process is used (note that$\xi_i^\sharp + \xi_i^\flat = \Delta Y_i$) $\Phi_{\Delta t}(P_{\Delta t}^*)$ is a perturbation, depending on the function h, of the semi-group $P_{\Delta t}^*$. With a suitable choice of $\Phi_{\Delta t}(\cdot)$, it is proved that :

$$\left\{E\int|\tilde{p}_i(x) - p(t_i,x)|^2\,dx\right\}^{1/2} \leq C\Delta t^{3/2}.$$

The interest of such product formulas is that the deterministic part and the stochastic part have been decoupled. In particular, the next steps (approximation of the semi-group $P_{\Delta t}^*$, and space-discretization) can be handled quite easily. Moreover, a probabilistic interpretation is available for the two numerical schemes described above.

4. Nonlinear filtering with high signal-to-noise ratio

Since the optimal filter is usually infinite dimensional, it is of practical importance to find good approximations in low dimension for certain classes of problems. One class of this kind is the class of problems with high signal-to-noise ratio, or in other words small observation noise, i.e. σ in (0.1) is "small". More precisely, we are looking for approximate finite dimensional filters, wich have a good behavior as $\sigma \to 0$.

This problem has been first considered by Katzur, Bobrovsky, Schuss [8] in the case where h is one-to-one. In that case, the filtering problem becomes trivial as $\sigma = 0$, i.e. the process $\{X_t\}$ is completely observed, and the variance of the conditional law is zero.

When $\sigma > 0$ is small, one may expect that the variance of the conditional law is small. Then, if $\hat{X}_t = E(X_t/\mathcal{Y}_t)$, X_t and $\hat{X}_t$ are close, and consequently :

$$\begin{aligned} f(X_t) &\simeq f(\hat{X}_t) + f'(\hat{X}_t)(X_t - \hat{X}_t) \\ g(X_t) &\simeq g(\hat{X}_t) \\ h(X_t) &\simeq h(\hat{X}_t) + h'(\hat{X}_t)(X_t - \hat{X}_t) \end{aligned}$$

But if we replace $f(X_t)$, $g(X_t)$ and $h(X_t)$ in (0.1) by their above approximations, we transform (0.1) into a linear filtering problem, which has a finite dimensional solution, namely the Kalman filter (which is the *extended* Kalman filter for (0.1)).

The above considerations tend to indicate that the extended Kalman filter (or possibly other types of approximate filters) might give good results as σ is small. This has been precisely formulated and rigorously proved by Picard [15] and then by Bensoussan [1] using a simpler argument.

Some new results have been obtained more recently.

4.a Nonlinear filtering with high signal-to-noise ratio in discrete time

Consider a discrete-time version of (0.1), i.e. :

$$
\begin{aligned}
X_{k+1} &= X_k + f(X_k)\,\Delta t + \sqrt{\Delta t}\,W_{k+1} \\
Y_k &= hX_k + \frac{\sigma}{\sqrt{\Delta t}}\,V_k
\end{aligned}
\tag{4.1}
$$

Note that we are in the case $g = 1$, h linear. If we let $\sigma \to 0$ while Δt is kept fixed, clearly there is no reason to use a more clever filter than the very simple estimate :

$$\hat{X}_k = h^{-1} Y_k$$

i.e. we use only the last observation. On the other hand, if σ and Δt tend to zero together, and we specify a relation of the form $\Delta t = \sigma^{\alpha}$ ($\alpha > 0$), one may expect to obtain a discrete time analog of Picard's result, i.e. one can show that the filter :

$$\hat{X}_{k+1} = \hat{X}_k + b(\hat{X}_k)\Delta t + \frac{\Delta t}{\sigma}(Y_{k+1} - hX_{k+1})$$

has a "good" behavior for $\sigma, \Delta t$ small.

This has been show in Milheiro [10]. This result will be very useful for numerical implementation of Picard's filters.

4.b Piecewise linear filtering with high signal-to-noise ratio

Suppose now that again σ is small, but now f and h are continuous and piecewise linear, while $g = 1$. If h is one-to-one, we can apply Picard's result. But we are specifically interested in the case where h is not one-to-one, i.e. for example :

$$h(x) = \begin{cases} h_- x, & \text{if } x \leq 0 \\ h_+ x, & \text{if } x \geq 0 \end{cases}$$

with $h_+ h_- < 0$. Suppose for simplicity that $f = 0$. If $h_+ = -h_-$, clearly the conditional law of X_t given $\mathcal{Y}_t$ is symetric around zero and there is no way to get a really good estimate of X_t. On the other hand, if $h_+ \neq -h_-$, for $\sigma = 0$ X_t is completely observed from $\{Y_s, 0 \leq s \leq t\}$ since the sign of X_t can be recovered from the quadratic variation of Y_t. Therefore, one may expect that, for σ small, the variance of the conditional law is small, and that $E(X_t/\mathcal{Y}_t)$ is well approximated by the output of the two Kalman filters corresponding to $h(x) = h_+ x$ and $h(x) = h_- x$.

Fleming, Ji, Pardoux [5] show that from the outputs of the two Kalman filters corresponding to $h(x) = h_+ x$ and $h(x) = h_- x$, one can define a test statistics, in order to decide the sign of X_t and consequently which of the two Kalman filters gives currently a good estimate of X_t. Note that the fact that one of a bank of Kalman filters gives a good estimate of X_t is true only in the case of a high signal-to-noise ratio. Without that hypothesis, a completely different approach has to be taken, see e.g. Pardoux, Savona [14], Savona [16].

5. Parameter estimation : The EM algorithm

We consider the following situation :

$$\begin{aligned} dX_t &= b_\theta(X_t)\,dt + \sigma(X_t)\,dW_t \\ dY_t &= h_\theta(X_t)\,dt + d\bar{W}_t \end{aligned}$$

with independant Wiener processes, and we assume that the law of X_0 has a density p_θ. The problem is to estimate the unknown parameter θ, on the basis of the observation of $\{Y_t\}$. It can be shown that the likelibood function for this problem can be computed using the solution of the corresponding Zakai equation. An alternative approach, the EM algorithm, has been considered by Dembo-Zeitouni [4] : it is an iterative algorithm where at each iteration, a new auxiliary function of the parameter is computed and maximized.

Campillo-Le Gland [2] have shown that the computation of this auxiliary function involves the solution of a nonlinear smoothing equation and also some recent results on stochastic integration with anticipating integrands due among others to Nualart and Pardoux.

Some numerical experiments have been made which show that, whenever some noise intensities in the system are small, the EM algorithm converges very slowly. Time discretizations of the stochastic partial differential involved have been proposed.

B) TRANSFER TO US

E. Pardoux has given a series of "distinguished lectures" at the Systems Research Center, Univ. of Maryland on the applications of the Malliavin calculus, and A. Bensoussan on Nonlinear filtering and stochastic control with partial observation.

F. Campillo has presented the results on the EM algorithm at the IEEE CDC in Los Angeles (December 1987).

E. Pardoux has presented some results on nonlinear filtering with high signal-to-noise ratio at the conference in the honor of W. Fleming, at Brown University (April 1988).

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MLE FOR PARTIALLY OBSERVED DIFFUSIONS: DIRECT MAXIMIZATION vs. THE EM ALGORITHM*

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Abstract

Two algorithms are compared for maximizing the likelihood function associated with parameter estimation in partially observed diffusion processes

- the EM algorithm, investigated by Dembo and Zeitouni [2], an iterative algorithm where, at each iteration, an auxiliary function is computed and maximized,
- the direct approach where the likelihood function itself is computed and maximized.

This yields to a comparison of nonlinear smoothing and nonlinear filtering for the computation of a class of conditional expectations related to the problem of estimation (Section 3). In particular, it is shown that smoothing is indeed necessary for the EM algorithm approach to be efficient.

Time–discretization schemes for the stochastic PDE's involved in the algorithms are given, and the link with the discrete–time case (hidden Markov model) is explored.

Numerical results are presented (Section 6) with the conclusion that direct maximization should be prefered whenever some noise covariances associated with the parameters to be estimated are small.

Keywords: *parameter estimation, maximum likelihood, EM algorithm, diffusion processes, nonlinear filtering, nonlinear smoothing, Skorokhod integral, time–discretization.*

Contents

1 Introduction: the EM algorithm

The EM algorithm is an iterative algorithm for maximizing a likelihood function, in a context of partial information [3]. Indeed, let $(P_\theta : \theta \in \Theta)$ be a family of mutually absolutely continuous probability measures on a measurable space $(\Omega, \mathcal{F})$, with $P_\theta \sim R$ and let $\mathcal{Y} \subset \mathcal{F}$ be the σ–algebra containing all the available information. Then, the log–likelihood function for the estimation of the parameter θ can be defined as

$$L(\theta) \triangleq \log \mathbf{E}_R \left(\frac{dP_\theta}{dR} \mid \mathcal{Y} \right) , \tag{1}$$

and the MLE (maximum likelihood estimate) as

$$\widehat{\theta} \in \arg\max_{\theta \in \Theta} L(\theta) .$$

The EM algorithm is based on the following straightforward application of Jensen's inequality

$$L(\theta) - L(\theta') = \log \mathbf{E}_{\theta'} \left(\frac{dP_\theta}{dP_{\theta'}} \mid \mathcal{Y} \right) \geq \mathbf{E}_{\theta'} \left(\log \frac{dP_\theta}{dP_{\theta'}} \mid \mathcal{Y} \right) \triangleq Q(\theta, \theta') , \tag{2}$$

which gives, for each value θ' of the parameter, a global minoration of the log–likelihood function $\theta \mapsto L(\theta)$ by means of an auxiliary function $\theta \mapsto L(\theta') + Q(\theta, \theta')$, with equality at $\theta = \theta'$. The algorithm iterations are described by the following steps

1. $p = 0$, initial guess $\widehat{\theta}_0$,
2. set $\theta' = \widehat{\theta}_p$,
3. (E–step) compute $Q(\cdot, \theta')$,
4. (M–step) find $\widehat{\theta}_{p+1}$ such that $Q(\widehat{\theta}_{p+1}, \theta') \geq Q(\theta, \theta')$ for all $\theta \in \Theta$,
5. if a stopping test is satisfied,
   then set final estimate $\theta^* = \widehat{\theta}_{p+1}$,
   else repeat from step 2 with $p = p + 1$.

An interesting feature of the algorithm is that it generates a maximizing sequence $\{\widehat{\theta}_p : p = 0, 1, \cdots\}$ in the sense that $L(\widehat{\theta}_{p+1}) > L(\widehat{\theta}_p)$ unless $\widehat{\theta}_{p+1} = \widehat{\theta}_p$. Some general convergence results about the sequences $\{L(\widehat{\theta}_p) : p = 0, 1, \cdots\}$ and $\{\widehat{\theta}_p : p = 0, 1, \cdots\}$ are proved in [13], under mild regularity assumptions on $L(\cdot)$ and $Q(\cdot, \cdot)$ – see also [2, Theorem 2]. To prove the existence of smooth enough – in the a.s. sense – versions of $\theta \mapsto L(\theta)$ and $(\theta, \theta') \mapsto Q(\theta, \theta')$, as well as to get the expression of the corresponding derivatives, one can rely e.g. on [12, Lemma 1].

To decide whether this algorithm is interesting from a computational point of view, the following three questions should be answered

[E] *how expensive is the computation of the auxiliary function $Q(\cdot, \theta')$?*

[M] *how easy is the maximization of the auxiliary function $Q(\cdot, \theta')$?*

[EM] *how fast is the convergence of this sub-optimal iterative algorithm towards the MLE ?*

In [2], the EM algorithm has been applied in the context of continuous–time partially observed stochastic processes. In the particular case of diffusion processes, the general expression of $Q(\theta,\theta')$ has been derived and said to involve a nonlinear smoothing problem. The purpose of this work is to address the following three points

- discuss the expression in [2] giving $Q(\theta,\theta')$ in terms of a nonlinear smoothing problem – this will involve generalized stochastic calculus (Skorokhod integral).
- get an equivalent expression, in terms of a nonlinear filtering problem, for $Q(\theta,\theta')$ and its gradient $\nabla^{1,0}Q(\theta,\theta')$ – it will turn out that smoothing is indeed necessary for the point [M] introduced above to be satisfied, although filtering is enough to compute $Q(\theta,\theta')$ for a given pair (θ,θ').
- get similar expressions for the original log–likelihood function $L(\theta)$ and its gradient $\nabla L(\theta)$.

This will allow to compare, from a computational point of view, the two possible methods for maximum likelihood estimation

- direct maximization of the likelihood function [4],
- the EM algorithm.

In particular, the point [M] will receive a positive answer, which is indeed the main motivation for the EM algorithm. On the other hand, it will be proved that computing the auxiliary function $Q(\cdot,\theta')$ is a more heavy task than computing the original log–likelihood function $L(\cdot)$. As for the point [EM], numerical examples will show that the convergence of the EM algorithm may be very slow. This typically occurs in those cases where, for each $\theta' \in \Theta$ the function $L(\theta')+Q(\cdot,\theta')$ is very sharp below the log–likelihood function $L(\cdot)$. In such cases indeed, maximizing the auxiliary function does not allow to update significantly enough the current estimate at each M–step.

The statistical model is presented in Section 2, where expressions are given for $L(\theta)$, $\nabla L(\theta)$, $Q(\theta,\theta')$ and $\nabla^{1,0}Q(\theta,\theta')$ in terms of conditional expectations. It turns out that the last three expressions all belong to a certain class of conditional expectations. Two methods are then proposed in Section 3 for the computation of conditional expectations in this class – one based on nonlinear filtering, the other on nonlinear smoothing and involving generalized stochastic calculus (Skorokhod integral). These results are applied in Section 4 to the computation of $L(\theta)$, $\nabla L(\theta)$, $Q(\theta,\theta')$ and $\nabla^{1,0}Q(\theta,\theta')$ in terms of nonlinear filtering and nonlinear smoothing conditional densities. Section 5 is devoted to the time–discretization of the stochastic PDE's introduced in Section 4, and the link with MLE of parameters in partially observed Markov chains (hidden Markov models) is explored. A numerical example is presented in Section 6, and the influence of noise covariances is investigated.

2 Statistical model

In this section, expressions for the log–likelihood function $L(\cdot)$ and the auxiliary function $Q(\cdot,\cdot)$ will be derived in the following context [2, Section 3].

Suppose that on a measurable space $(\Omega, \mathcal{F})$ are given

- a family $(P_\theta : \theta \in \Theta)$ of probability measures,
- a pair of stochastic processes $(X_t : t \geq 0)$ and $(Y_t : t \geq 0)$ taking values in $\mathbf{R}^m$ and $\mathbf{R}^d$ respectively,

such that under P_θ

$$\begin{aligned} dX_t &= b_\theta(X_t)\,dt + \sigma(X_t)\,dW_t^\theta\,, \qquad X_0 \sim p_0^\theta(\cdot)\,, \\ dY_t &= h_\theta(X_t)\,dt + d\overline{W}_t^\theta\,, \end{aligned} \tag{3}$$

where $(W_t^\theta : t \geq 0)$ and $(\overline{W}_t^\theta : t \geq 0)$ are independent Wiener processes, with covariance matrix I and r respectively, and the pair is independent from the r.v. X_0.

The following hypotheses are made

(H_1) *$\sigma(\cdot)$ is a continuous and bounded function on $\mathbf{R}^m$ such that $a(\cdot) \triangleq \sigma\sigma^*(\cdot)$ is a uniformly elliptic $m \times m$ matrix, i.e. $a(\cdot) \geq \alpha I$,*

for all $\theta \in \Theta$ open subset of $\mathbf{R}^p$ (the set of parameters)

(H_2) *$p_0^\theta(\cdot)$ is a density on $\mathbf{R}^m$,*

(H_3) *$b_\theta(\cdot)$ is a measurable and bounded function from $\mathbf{R}^m$ to $\mathbf{R}^m$,*

(H_4) *$h_\theta(\cdot)$ is a measurable and bounded function from $\mathbf{R}^m$ to $\mathbf{R}^d$,*

and in addition

(H_5) *the probability measures on $\mathbf{R}^m$ with densities $(p_0^\theta(\cdot) : \theta \in \Theta)$ are mutually absolutely continuous.*

Moreover, it is assumed that $p_0^\theta(\cdot)$, $b_\theta(\cdot)$ and $h_\theta(\cdot)$ are continuously differentiable with respect to the parameter θ and that, for all $\theta \in \Theta$ the derivatives $\nabla b_\theta(\cdot)$ and $\nabla h_\theta(\cdot)$ are measurable and bounded functions from $\mathbf{R}^m$ to $\mathbf{R}^m$ and $\mathbf{R}^d$ respectively (throughout this paper, ∇ will denote the derivation with respect to the parameter θ).

The existence and uniqueness of a weak solution to the stochastic differential equation (3) follows from hypotheses $(H_1 - H_3)$. If moreover hypotheses $(H_4 - H_5)$ hold, then for all $T \geq 0$, $(P_\theta : \theta \in \Theta)$ when restricted to $[0, T]$ are mutually absolutely continuous probability measures on $(\Omega, \mathcal{F})$ with Radon–Nikodym derivative

$$\Lambda_{\theta\theta'} \triangleq \frac{dP_\theta}{dP_{\theta'}} =$$

$$= \frac{p_0^\theta}{p_0^{\theta'}}(X_0) \cdot \exp\left\{ \int_0^T [b_\theta(X_s) - b_{\theta'}(X_s)]^* a^{-1}(X_s)\sigma(X_s)\, dW_s^{\theta'} \right. \tag{4}$$
$$\left. -\tfrac{1}{2}\int_0^T [b_\theta(X_s) - b_{\theta'}(X_s)]^* a^{-1}(X_s)[b_\theta(X_s) - b_{\theta'}(X_s)]\, ds \right\}$$
$$\exp\left\{ \int_0^T [h_\theta(X_s) - h_{\theta'}(X_s)]^* r^{-1}\, d\overline{W}_s^{\theta'} \right.$$
$$\left. -\tfrac{1}{2}\int_0^T [h_\theta(X_s) - h_{\theta'}(X_s)]^* r^{-1}[h_\theta(X_s) - h_{\theta'}(X_s)]\, ds \right\} .$$

Consider also the probability measure $P_\theta^\dagger$ defined on $(\Omega, \mathcal{F})$ by

$$Z^\theta \triangleq \frac{dP_\theta}{dP_\theta^\dagger} = \exp\left\{ \int_0^T h_\theta^*(X_s) r^{-1}\, dY_s - \tfrac{1}{2}\int_0^T h_\theta^*(X_s) r^{-1} h_\theta(X_s)\, ds \right\} ,$$

so that, under $P_\theta^\dagger$

$$dX_t = b_\theta(X_t)\, dt + \sigma(X_t)\, dW_t^\theta , \qquad X_0 \sim p_0^\theta(\cdot) ,$$

where $(W_t^\theta : t \geq 0)$ and $(Y_t^\theta : t \geq 0)$ are independent Wiener processes, with covariance matrix I and r respectively, and the pair is independent from the r.v. X_0. The Radon–Nikodym derivative $\Lambda_{\theta\theta'}$ can then be decomposed as

$$\Lambda_{\theta\theta'} = \Lambda_{\theta\theta'}^\dagger \frac{Z^\theta}{Z^{\theta'}} , \qquad \text{with} \qquad \Lambda_{\theta\theta'}^\dagger \triangleq \frac{dP_\theta^\dagger}{dP_{\theta'}^\dagger} .$$

It is assumed that only $(Y_t : 0 \leq t \leq T)$ is observed. Let $(\mathcal{Y}_t : 0 \leq t \leq T)$ denote the associated filtration. The likelihood function for the estimation of the parameter θ can be expressed as

$$\mathbf{E}_\alpha^\dagger \left(\frac{dP_\theta}{dP_\alpha^\dagger} \mid \mathcal{Y}_T \right) = \mathbf{E}_\alpha^\dagger (Z^\theta \Lambda_{\theta\alpha}^\dagger \mid \mathcal{Y}_T)$$

with the particular choice $R = P_\alpha^\dagger$ (α fixed in Θ) in (1). By Bayes formula

$$\mathbf{E}_\alpha^\dagger (Z^\theta \Lambda_{\theta\alpha}^\dagger \mid \mathcal{Y}_T) = \mathbf{E}_\theta^\dagger (Z^\theta \mid \mathcal{Y}_T) \cdot \mathbf{E}_\alpha^\dagger (\Lambda_{\theta\alpha}^\dagger \mid \mathcal{Y}_T) = \mathbf{E}_\theta^\dagger (Z^\theta \mid \mathcal{Y}_T)$$

since $\Lambda_{\theta\alpha}^\dagger$ is independent of $\mathcal{Y}_T$ under $P_\alpha^\dagger$. This gives the following expression for the log–likelihood function $L(\cdot)$

$$L(\theta) = \log \mathbf{E}_\theta^\dagger (Z^\theta \mid \mathcal{Y}_T) . \tag{5}$$

For the auxiliary function $Q(\cdot,\cdot)$ defined by (2), one has immediately

$$Q(\theta, \theta') = \mathbf{E}_{\theta'}(\lambda^{\theta,\theta'} \mid \mathcal{Y}_T) = \frac{\mathbf{E}_{\theta'}^\dagger (\lambda^{\theta,\theta'} Z^{\theta'} \mid \mathcal{Y}_T)}{\mathbf{E}_{\theta'}^\dagger (Z^{\theta'} \mid \mathcal{Y}_T)} \tag{6}$$

where

$$\lambda^{\theta,\theta'} \triangleq \log \Lambda_{\theta,\theta'}$$

$$
\begin{aligned}
= \log \frac{p_0^\theta}{p_0^{\theta'}}(X_0) &+ \int_0^T [b_\theta(X_s) - b_{\theta'}(X_s)]^* a^{-1}(X_s)\sigma(X_s)\, dW_s^{\theta'} \\
&-\tfrac{1}{2}\int_0^T [b_\theta(X_s) - b_{\theta'}(X_s)]^* a^{-1}(X_s)[b_\theta(X_s) - b_{\theta'}(X_s)]\, ds \\
&+ \int_0^T [h_\theta(X_s) - h_{\theta'}(X_s)]^* r^{-1}\, d\overline{W}_s^{\theta'} \\
&-\tfrac{1}{2}\int_0^T [h_\theta(X_s) - h_{\theta'}(X_s)]^* r^{-1}[h_\theta(X_s) - h_{\theta'}(X_s)]\, ds \ .
\end{aligned} \tag{7}
$$

Under additional regularity asumptions on the data $p_0^\theta(\cdot)$, $b_\theta(\cdot)$ and $h_\theta(\cdot)$, it is easy to prove, using results in [12], that both $\theta \mapsto L(\theta)$ and $\theta \mapsto Q(\theta,\theta')$ have a.s. differentiable versions, with gradients given by

$$
\nabla L(\theta) \ = \ \mathbf{E}_\theta(\lambda^\theta \mid \mathcal{Y}_T) = \frac{\mathbf{E}_\theta^\dagger(\lambda^\theta Z^\theta \mid \mathcal{Y}_T)}{\mathbf{E}_\theta^\dagger(Z^\theta \mid \mathcal{Y}_T)} \ , \tag{8}
$$

$$
\nabla^{1,0} Q(\theta,\theta') \ = \ \mathbf{E}_{\theta'}(\lambda^\theta \mid \mathcal{Y}_T) = \frac{\mathbf{E}_{\theta'}^\dagger(\lambda^\theta Z^\theta \mid \mathcal{Y}_T)}{\mathbf{E}_{\theta'}^\dagger(Z^\theta \mid \mathcal{Y}_T)} \ , \tag{9}
$$

respectively, where

$$
\begin{aligned}
\lambda^\theta &\triangleq \nabla^{1,0} \log \Lambda_{\theta,\theta'} = \nabla^{1,0} \lambda^{\theta,\theta'} \\
&= \frac{\nabla p_0^\theta}{p_0^\theta}(X_0) + \int_0^T [\nabla b_\theta(X_s)]^* a^{-1}(X_s)\sigma(X_s)\, dW_s^\theta + \int_0^T [\nabla h_\theta(X_s)]^* r^{-1}\, d\overline{W}_s^\theta
\end{aligned} \tag{10}
$$

is independent of θ'.

Remark. One can check from (8) and (9) that

$$
\nabla^{1,0} Q(\theta,\theta') \mid_{\theta=\theta'} = \nabla L(\theta') \ ,
$$

as expected.

In the next section, two different methods will be given – by means of stochastic PDE's – to compute the various quantities introduced so far: $L(\theta)$, $\nabla L(\theta)$, $Q(\theta,\theta')$ and $\nabla^{1,0} Q(\theta,\theta')$. This will make possible the numerical implementation of algorithms for the maximization of the likelihood function. ∎

3 Smoothing vs. filtering for the computation of a class of conditional expectations

For the sake of simplicity, any reference to the parameter θ will be dropped throughout this section. In particular, P will denote the probability measure under which

$$\begin{aligned} dX_t &= b(X_t)\,dt + \sigma(X_t)\,dW_t\ , \qquad X_0 \sim p_0(\cdot)\ , \\ dY_t &= h(X_t)\,dt + d\overline{W}_t\ , \end{aligned}$$

where $(W_t\ :\ 0 \leq t \leq T)$ and $(\overline{W}_t\ :\ 0 \leq t \leq T)$ are independent Wiener processes, with covariance matrix I and r respectively, and the pair is independent from the r.v. X_0, whereas under $P^\dagger$

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t\ , \qquad X_0 \sim p_0(\cdot)\ ,$$

where $(W_t\ :\ 0 \leq t \leq T)$ and $(Y_t\ :\ 0 \leq t \leq T)$ are independent Wiener processes, with covariance matrix I and r respectively, and the pair is independent from the r.v. X_0. Therefore $P = Z_T \cdot P^\dagger$, where the process $(Z_t\ :\ 0 \leq t \leq T)$ is defined by

$$Z_t = \exp\left\{\int_0^t h^*(X_s) r^{-1}\,dY_s - \tfrac{1}{2}\int_0^t h^*(X_s) r^{-1} h(X_s)\,ds\right\}\ .$$

The purpose of this section is to provide two different methods – one based on nonlinear filtering, the other on nonlinear smoothing – for the computation of the following class of conditional expectations

$$A \triangleq \mathbf{E}\left(\beta(X_0) + \int_0^T \xi(X_s)\,ds + \int_0^T \eta^*(X_s)\,d\overline{W}_s + \int_0^T \chi^*(X_s)\sigma(X_s)\,dW_s \mid \mathcal{Y}_T\right)\ , \qquad (11)$$

where β, ξ, η and χ are measurable and bounded functions from $\mathbf{R}^m$ to $\mathbf{R}$, $\mathbf{R}$, $\mathbf{R}^d$ and $\mathbf{R}^m$ respectively. It is readily seen from (6–10) that the computation of either $\nabla L(\theta)$, $Q(\theta,\theta')$ or $\nabla^{1,0}Q(\theta,\theta')$ involves such conditional expectations.

It is clear from the definition that A depends linearly on (β,ξ,η,χ). It will turn out that nonlinear smoothing is the only way to make this dependence explicit, although nonlinear filtering – which is simpler – is enough to just compute A.

Rewriting A as

$$\begin{aligned} A &= \mathbf{E}(\beta(X_0) \mid \mathcal{Y}_T) + \int_0^T \mathbf{E}(\xi(X_s) - \eta^*(X_s)h(X_s) \mid \mathcal{Y}_T)\,ds + \mathbf{E}\left(\int_0^T \eta^*(X_s)\,dY_s \mid \mathcal{Y}_T\right) \\ &\quad + \mathbf{E}\left(\int_0^T \chi^*(X_s)\sigma(X_s)\,dW_s \mid \mathcal{Y}_T\right)\ , \end{aligned} \qquad (12)$$

one would like to interchange conditional expectation and stochastic integral in the third term of (12). However, the resulting expression

$$\text{“}\int_0^T \mathbf{E}(\eta^*(X_s) \mid \mathcal{Y}_T)\,dY_s\text{”} \qquad (13)$$

is not an Itô integral, since the integrand is obviously not adapted to the filtration $(\mathcal{Y}_t : 0 \leq t \leq T)$, and needs to be given a rigorous meaning. Although the natural generalization of Itô integral that allows anticipating integrands is Skorokhod integral, it will be proved in Proposition 3.3 below that the correct statement is

$$\mathbf{E}\left(\int_0^T \eta^*(X_s)\,dY_s \mid \mathcal{Y}_T\right) = \mathbf{E}\left(\int_0^T \eta^*(X_s) \circ dY_s \mid \mathcal{Y}_T\right)$$
$$= \int_0^T \mathbf{E}(\eta^*(X_s) \mid \mathcal{Y}_T) \circ dY_s \neq \int_0^T \mathbf{E}(\eta^*(X_s) \mid \mathcal{Y}_T)\,dY_s \,,$$

where the non–adapted stochastic integrals are respectively a generalized Stratonovitch integral and a Skorokhod integral [6].

In addition, there seems to be no computable expression available for the last term of (12). However, in the particular case where χ derives from a scalar potential function, one has the following

Proposition 3.1. *Assume there exists a scalar function $U \in C_b^2(\mathbf{R}^m)$ such that $\chi = DU$. Then*

$$\mathbf{E}\left(\int_0^T \chi^*(X_s)\sigma(X_s)\,dW_s \mid \mathcal{Y}_T\right) =$$
$$\mathbf{E}(U(X_T) \mid \mathcal{Y}_T) - \mathbf{E}(U(X_0) \mid \mathcal{Y}_T) - \int_0^T \mathbf{E}(\mathcal{L}U(X_s) \mid \mathcal{Y}_T)\,ds \,, \tag{14}$$

whose proof follows immediately from Itô's lemma.

At this point, it is necessary to introduce some notations and definitions related to nonlinear filtering and smoothing.

Notations and definitions

- *Filtering*

π_t (resp. p_t) will denote the normalized (resp. unnormalized) conditional density of the r.v. X_t given $\mathcal{Y}_t$, i.e.

$$(\pi_t, \phi) \triangleq \mathbf{E}(\phi(X_t) \mid \mathcal{Y}_t) \,, \qquad (p_t, \phi) \triangleq \mathbf{E}^\dagger(\phi(X_t)Z_t \mid \mathcal{Y}_t) \tag{15}$$

for any test–function ϕ. By Bayes formula

$$(\pi_t, \phi) = \frac{(p_t, \phi)}{(p_t, 1)} \,. \tag{16}$$

The equation for $(p_t : 0 \leq t \leq T)$ is Zakaï equation [8]

$$dp_t = \mathcal{L}^* p_t\,dt + h^* p_t r^{-1}\,dY_t \,, \tag{17}$$

where $\mathcal{L}^*$ denotes the adjoint operator of the infinitesimal generator $\mathcal{L}$ of the diffusion process $(X_t : 0 \leq t \leq T)$, defined by

$$\mathcal{L} \triangleq \tfrac{1}{2} \sum_{i,j=1}^m a^{ij}(\cdot) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^m b^i(\cdot) \frac{\partial}{\partial x_i} \,.$$

• *Smoothing (fixed–interval)*

Let $T > 0$ denote the fixed end–time. ρ_t (resp. q_t) will denote the normalized (resp. unnormalized) conditional density of the r.v. X_t given $\mathcal{Y}_T$, i.e.

$$(\rho_t, \phi) \triangleq \mathbf{E}(\phi(X_t) \mid \mathcal{Y}_T) , \qquad (q_t, \phi) \triangleq \mathbf{E}^\dagger(\phi(X_t) Z_T \mid \mathcal{Y}_T) .$$

Again

$$(\rho_t, \phi) = \frac{(q_t, \phi)}{(q_t, 1)} . \tag{18}$$

Introducing the backward Zakaï equation

$$dv_t + \mathcal{L} v_t \, dt + h^* v_t r^{-1} \, dY_t = 0 , \qquad v_T \equiv 1 , \tag{19}$$

one has [8,9] that (p_t, v_t) is independent of t, and $q_t = p_t v_t$ is differentiable with

$$\dot{q}_t + p_t \, \mathcal{L} v_t = v_t \, \mathcal{L}^* p_t . \tag{20}$$

Note that

$$(q_t, 1) = (p_T, 1) , \quad 0 \le t \le T . \tag{21}$$

3.1 Filtering approach

Define

$$\lambda_t \triangleq \beta(X_0) + \int_0^t \xi(X_s) \, ds + \int_0^t \eta^*(X_s) \, d\overline{W}_s + \int_0^t \chi^*(X_s) \sigma(X_s) \, dW_s$$

so that, by Bayes formula

$$A = \mathbf{E}(\lambda_T \mid \mathcal{Y}_T) = \frac{\mathbf{E}^\dagger(\lambda_T Z_T \mid \mathcal{Y}_T)}{\mathbf{E}^\dagger(Z_T \mid \mathcal{Y}_T)} .$$

A first method would be to compute the joint conditional law of (X_T, λ_T) given $\mathcal{Y}_T$, and then integrate over the first variable to get the marginal conditional law of λ_T given $\mathcal{Y}_T$. An alternative method is to find an equation for $(w_t : 0 \le t \le T)$ defined by

$$(w_t, \phi) \triangleq \mathbf{E}^\dagger(\phi(X_t) \lambda_t Z_t \mid \mathcal{Y}_t) .$$

Indeed, by Itô's lemma

$$\begin{aligned} d[\phi(X_t)\lambda_t Z_t] = {} & \lambda_t \, Z_t \, \mathcal{L}\phi(X_t) \, dt + \lambda_t \, Z_t \, (D\phi(X_t))^* \, \sigma(X_t) \, dW_t \\ & + \phi(X_t) \, Z_t \, \xi(X_t) \, dt + \phi(X_t) \, Z_t \, \eta^*(X_t) \, d\overline{W}_t + \phi(X_t) \, Z_t \, \chi^*(X_t) \, \sigma(X_t) \, dW_t \\ & + \phi(X_t) \, \lambda_t \, Z_t \, h^*(X_t) \, r^{-1} \, dY_t + \phi(X_t) \, \eta^*(X_t) \, h(X_t) \, Z_t \, dt \\ & + Z_t \, \chi(X_t)^* \, a(X_t) \, D\phi(X_t) \, dt . \end{aligned}$$

Using properties of conditional expectation given the observation σ–algebra under the reference probability measure $P^\dagger$, and the definition (15), gives

$$
\begin{aligned}
(w_t,\phi) \;=\; & (p_0,\beta\phi) + \int_0^t (w_s,\mathcal{L}\phi)\,ds + \int_0^t (w_s,h^*\phi)r^{-1}\,dY_s \\
& + \int_0^t (p_s,\xi\phi)\,ds + \int_0^t (p_s,\eta^*\phi)\,dY_s + \int_0^t (p_s,\mathcal{J}(\chi)\phi)\,ds\ ,
\end{aligned}
$$

where $\mathcal{J}(\chi)\phi \triangleq \chi^* a D\phi$, so that $(w_t\,:\,0\le t\le T)$ solves

$$
dw_t = \mathcal{L}^* w_t\,dt + h^* w_t r^{-1}\,dY_t + \xi p_t\,dt + \eta^* p_t\,dY_t + \mathcal{J}^*(\chi)p_t\,dt\ , \qquad w_0 = \beta p_0\ . \tag{22}
$$

Theorem 3.2. *Let $(p_t\,:\,0\le t\le T)$ and $(w_t\,:\,0\le t\le T)$ be the unique solution of (17) and (22) respectively. Then, the following expression holds for A defined in (11)*

$$
A = \frac{(w_T,1)}{(p_T,1)}\ . \tag{23}
$$

This expression is actually computable. Unfortunately, the linear dependence of $(w_T,1)$ on (β,ξ,η,χ) is not made explicit, which should be the case for the point [M] introduced in the Introduction to be satisfied. Therefore, the next step will be to make this dependence more explicit. This will involve nonlinear smoothing and generalized stochastic calculus (Skorokhod integral). Actually

- the stochastic integral in (13) will be given a rigorous meaning,
- the last term in (12) will also be given a computable expression, whether or not χ derives from a scalar potential function.

3.2 Smoothing approach

The idea here is to compute the stochastic differential of the scalar product (w_t,v_t), where $(v_t\,:\,0\le t\le T)$ is the solution of the backward Zakaï equation (19). Since (22) is a forward stochastic PDE and (19) is a backward stochastic PDE, one must use the two–sided stochastic calculus introduced in [10,11]. This gives

$$
\begin{aligned}
d\,(w_t,v_t) \;=\; & (\mathcal{L}^* w_t,v_t)\,dt + (h^* w_t,v_t)r^{-1}\,dY_t \\
& + (\xi p_t,v_t)\,dt + (\eta^* p_t,v_t)\,dY_t + (\mathcal{J}^*(\chi)p_t,v_t)\,dt \\
& - (w_t,\mathcal{L}v_t)\,dt - (w_t,h^* v_t)r^{-1}\,dY_t \\
\;=\; & (q_t,\xi)\,dt + (q_t,\eta^*)\,dY_t + (p_t,\chi^* a\,Dv_t)\,dt\ .
\end{aligned}
$$

Integrating from 0 to T gives

$$
(w_T,1) = (q_0,\beta) + \int_0^T (q_s,\xi)\,ds + \int_0^T (q_s,\eta^*)\,dY_s + \int_0^T (p_s,\chi^* a\,Dv_s)\,ds\ ,
$$

where the stochastic integral is a two–sided stochastic integral.

Using (21) gives an expression for A in terms of normalized conditional densities

$$
A = (\rho_0,\beta) + \int_0^T (\rho_s,\xi)\,ds + A' + A''\ .
$$

• *Study of A'*

$$A' \triangleq \frac{\int_0^T (q_s, \eta^*)\, dY_s}{(p_T, 1)} \ . \tag{24}$$

One has

$$\begin{aligned} \mathbf{E}(A') &= \mathbf{E}^\dagger(Z_T\, A') = \mathbf{E}^\dagger(\mathbf{E}^\dagger(Z_T \mid \mathcal{Y}_T)\, A') \\ &= \mathbf{E}^\dagger \left(\int_0^T (q_s, \eta^*)\, dY_s \right) = 0 \ , \end{aligned}$$

where the last equality follows from results on two–sided stochastic integrals. This was expected, since

$$A' = \mathbf{E} \left(\int_0^T \eta^*(X_s)\, d\overline{W}_s \mid \mathcal{Y}_T \right) \ .$$

Expressions in terms of normalized conditional densities are given by the following

Proposition 3.3. *Let $(\rho_t : 0 \le t \le T)$ denote the normalized smooothing density. Then*

$$\begin{aligned} A' &= \int_0^T (\rho_s, \eta^*)\, dY_s - \int_0^T (\rho_s, \eta^*)\, (\rho_s, h)\, ds \\ &= \int_0^T (\rho_s, \eta^*) \circ dY_s - \int_0^T (\rho_s, \eta^* h)\, ds \ , \end{aligned}$$

where the non–adapted stochastic integrals are respectively a Skorokhod integral and a generalized Stratonovitch integral [6].

PROOF. The idea is to get the denominator $F \triangleq 1/(p_T, 1)$ inside the stochastic integral in (24).

Let first $D.$ denote, on the probability space $(\Omega, \mathcal{Y}_T, P^\dagger)$, the derivative with respect to the d–dimensional Wiener process $(Y_t : 0 \le t \le T)$ in the direction of the vector space $H^1(0, T; \mathbf{R}^d)$. Since the two–sided integral is a particular case of the Skorokhod integral, it follows from [6, Proposition 3.2] that

$$\begin{aligned} A' &= F \int_0^T (q_s, \eta^*)\, dY_s \\ &= \int_0^T F\, (q_s, \eta^*)\, dY_s + \int_0^T (q_s, \eta^*)\, \underset{s}{D_\cdot} F\, ds \\ &= \int_0^T \frac{(q_s, \eta^*)}{(q_s, 1)}\, dY_s - \int_0^T \frac{(q_s, \eta^*)}{(q_s, 1)^2}\, D_s(p_T, 1)\, ds \ , \end{aligned}$$

where the stochastic integral is a Skorokhod integral.

For s fixed in $[0, T]$, consider the d–dimensional random process $(z_t : 0 \le t \le T)$ defined by $z_t \triangleq D_s p_t$. Clearly $z_t \equiv 0$ for $0 \le t < s$. For $i = 1, \cdots, d$, the process $(z_t^i : s \le t \le T)$ is the unique solution of the forward stochastic PDE [7]

$$dz_t^i = \mathcal{L}^* z_t^i\, dt + h^* z_t^i r^{-1}\, dY_t \ , \qquad z_s^i = h^i p_s \ .$$

Introducing the solution $(v_t : 0 \leq t \leq T)$ of the backward Zakaï equation (19) and using again the two-sided stochastic calculus gives $d(z_t, v_t) = 0$ for $s \leq t \leq T$. Therefore

$$(z_T, 1) = (z_s, v_s) = (hp_s, v_s) = (q_s, h) \ ,$$

so that

$$\begin{aligned} A' &= \int_0^T \frac{(q_s, \eta^*)}{(q_s, 1)} \, dY_s - \int_0^T \frac{(q_s, \eta^*)}{(q_s, 1)} \frac{(q_s, h)}{(q_s, 1)} \, ds \\ &= \int_0^T (\rho_s, \eta^*) \, dY_s - \int_0^T (\rho_s, \eta^*) (\rho_s, h) \, ds \ . \end{aligned}$$

To get the second expression, consider the d–dimensional random process $(u_t : 0 \leq t \leq T)$ defined by $u_t \triangleq (\rho_t, \eta)$. The Skorokhod–Stratonovitch transformation for generalized stochastic integrals gives [6, Theorem 7.3]

$$\int_0^T u_s^* \, dY_s = \int_0^T u_s^* \circ dY_s - \tfrac{1}{2} \int_0^T (D_s^+ u_s + D_s^- u_s) \, ds \ ,$$

where

$$D_s^+ u_s \triangleq \lim_{t \downarrow s} \sum_{i=1}^d D_s^i u_t^i \ , \qquad D_s^- u_s \triangleq \lim_{t \uparrow s} \sum_{i=1}^d D_s^i u_t^i \ .$$

It turns out that

$$\begin{aligned} D_s^i u_t^i &= D_s^i (\rho_t, \eta^i) = D_s^i \frac{(q_t, \eta^i)}{(q_t, 1)} \\ &= \frac{D_s^i (q_t, \eta^i)}{(q_t, 1)} - \frac{(q_t, \eta^i) \, D_s^i (q_t, 1)}{(q_t, 1)^2} \ . \end{aligned}$$

Next $D_s q_t = (D_s p_t) \, v_t + p_t \, (D_s v_t)$. In particular $D_s p_t$ has already been studied, and a similar argument for $D_s v_t$ shows that $D_s^+ q_s = D_s^- q_s = h q_s$. Therefore

$$\begin{aligned} D_s^+ u_s = D_s^- u_s &= \frac{(q_s, \eta^* h)}{(q_s, 1)} - \frac{(q_t, \eta^*) (q_t, h)}{(q_t, 1)^2} \\ &= (\rho_s, \eta^* h) - (\rho, \eta^*) (\rho_s, h) \ . \end{aligned}$$

This finaly gives

$$A' = \int_0^T (\rho_s, \eta^*) \circ dY_s - \int_0^T (\rho_s, \eta^* h) \, ds \ ,$$

where the stochastic integral is now a generalized Stratonovitch integral. □

Remark. In terms of conditional expectations

$$\begin{aligned} A' &= \int_0^T \mathbf{E}(\eta^*(X_s) \mid \mathcal{Y}_T) \, dY_s - \int_0^T \mathbf{E}(\eta^*(X_s) \mid \mathcal{Y}_T) \, \mathbf{E}(h(X_s) \mid \mathcal{Y}_T) \, ds \\ &= \int_0^T \mathbf{E}(\eta^*(X_s) \mid \mathcal{Y}_T) \circ dY_s - \int_0^T \mathbf{E}(\eta^*(X_s) h(X_s) \mid \mathcal{Y}_T) \, ds \ . \end{aligned}$$

• *Study of A''*

$$A'' \triangleq \frac{\int_0^T (p_s, \chi^* a\, Dv_s)\, ds}{(p_T, 1)} . \tag{25}$$

One has

$$\begin{aligned} \mathbf{E}(A'') &= \mathbf{E}^\dagger(Z_T\, A'') = \mathbf{E}^\dagger(\mathbf{E}^\dagger(Z_T \mid \mathcal{Y}_T)\, A'') \\ &= \mathbf{E}^\dagger\left(\int_0^T (p_s, \chi^* a\, Dv_s)\, ds\right) = \int_0^T (\mathbf{E}^\dagger(p_s), \chi^* a\, \mathbf{E}^\dagger(Dv_s))\, ds , \end{aligned}$$

where the last equality follows from the independence of p_s and v_s under the probability measure $P^\dagger$. Now $\mathbf{E}^\dagger(Dv_s) = D\mathbf{E}^\dagger(v_s) \equiv 0$ since $\mathbf{E}^\dagger(v_s) \equiv 1$. Therefore $\mathbf{E}(A'') = 0$, which was expected since

$$A'' = \mathbf{E}\left(\int_0^T \chi^*(X_s)\sigma(X_s)\, dW_s \mid \mathcal{Y}_T\right) .$$

The identities

$$\frac{p_s\, Dv_s}{(q_s, 1)} = \pi_s\, D\left(\frac{\rho_s}{\pi_s}\right) = \rho_s\, D\left(\log \frac{\rho_s}{\pi_s}\right)$$

give the following two other expressions for A'', in terms of normalized conditional densities

$$A'' = \int_0^T (\pi_s, \chi^* a\, D\left(\frac{\rho_s}{\pi_s}\right))\, ds = \int_0^T (\rho_s, \chi^* a\, D\left(\log \frac{\rho_s}{\pi_s}\right))\, ds .$$

In the particular case where χ derives from a scalar potential function, it can be checked that (25) reduces to the expression (14) given in Proposition 3.1. Indeed

Proposition 3.4. *Assume there exists a scalar function $U \in C_b^2(\mathbf{R}^m)$ such that $\chi = DU$. Then*

$$A'' = (\rho_T, U) - (\rho_0, U) - \int_0^T (\rho_s, \mathcal{L}U)\, ds .$$

Proof. It follows from the identity $\mathcal{L}(Uv_s) = U\mathcal{L}v_s + v_s\mathcal{L}U + \chi^* a Dv_s$, and from (20 that

$$\begin{aligned} (p_s, \chi^* a Dv_s) &= (p_s, \mathcal{L}(Uv_s)) - (p_s, U\mathcal{L}v_s) - (p_s, v_s\mathcal{L}U) \\ &= (v_s\mathcal{L}^* p_s - p_s\mathcal{L}v_s, U) - (p_s v_s, \mathcal{L}U) = (\dot{q}_s, U) - (q_s, \mathcal{L}U) . \end{aligned}$$

Integrating from 0 to T gives

$$\int_0^T (p_s, \chi^* a\, Dv_s)\, ds = (q_T, U) - (q_0, U) - \int_0^T (q_s, \mathcal{L}U)\, ds .$$

Dividing by $(p_T, 1)$ and using (21) finishes the proof. □

Remark. In terms of conditional expectations

$$A'' = \mathbf{E}(U(X_T) \mid \mathcal{Y}_T) - \mathbf{E}(U(X_0) \mid \mathcal{Y}_T) - \int_0^T \mathbf{E}(\mathcal{L}U(X_s) \mid \mathcal{Y}_T)\, ds ,$$

which is exactly (14).

The following theorem has been proved

Theorem 3.5. *Let $(\pi_t : 0 \le t \le T)$ and $(\rho_t : 0 \le t \le T)$ be the normalized filtering and smoothing density (e.g. obtained from the unique solution $(p_t : 0 \le t \le T)$ and $(v_t : 0 \le t \le T)$ of (17) and (19) respectively). Then, the following two expressions hold for A defined in (11)*

$$A = (\rho_0, \beta) + \int_0^T (\rho_s, \xi)\, ds + \int_0^T \left(\rho_s, \chi^* a\, D\left(\log \frac{\rho_s}{\pi_s}\right)\right) ds + A' \ ,$$

$$A' = \begin{cases} \displaystyle\int_0^T (\rho_s, \eta^*)\, dY_s - \int_0^T (\rho_s, \eta^*)\,(\rho_s, h)\, ds \ , \\[2ex] \displaystyle\int_0^T (\rho_s, \eta^*) \circ dY_s - \int_0^T (\rho_s, \eta^* h)\, ds \ , \end{cases}$$

where the non-adapted stochastic integrals are respectively a Skorokhod integral and a generalized Stratonovitch integral [6].

Conclusion

The advantage of smoothing over filtering is that the linear dependence on $(\beta,\ \xi,\ \eta,\ \chi)$ is made explicit: provided the underlying probability measure does not change, evaluating A for a different set of data (β, ξ, η, χ) will not require the computation of a new infinite–dimensional conditional density. In the filtering approach, one would have to solve another stochastic PDE, with a different "right–hand side".

On the other hand, from the computational point of view, solving the equation for the smoothing density requires not only the computation but also the storage of the filtering density, and is therefore more expensive. Moreover, in the filtering approach it is enough to integrate the unnormalized filtering density at final time T, whereas in the smoothing approach one has *(i)* at each time t, to integrate some functions involving (ξ, η, χ) against the normalized smoothing density, and *(ii)* to integrate the resulting processes over the interval $[0, T]$.

The next section will be devoted to applying these two approaches to the computation of quantities related to the direct likelihood function maximization and the EM algorithm.

4 Application to the MLE problem

4.1 Direct maximization of the likelihood function

It follows from (5) that the log-likelihood function $L(\theta)$ can be expressed as

$$L(\theta) = \log(p^\theta_T, 1)$$

with – see (17)

$$dp^\theta_t = \mathcal{L}^*_\theta p^\theta_t \, dt + h^*_\theta p^\theta_t r^{-1} \, dY_t \tag{26}$$

and

$$\mathcal{L}_\theta \triangleq \tfrac{1}{2} \sum_{i,j=1}^{m} a^{ij}(\cdot) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b^i_\theta(\cdot) \frac{\partial}{\partial x_i} \, .$$

It follows from (8) and (10) that $\nabla L(\theta)$ belongs to the class of conditional expectations considered in Section 3. The approach based on filtering (Theorem 3.2) gives

$$\nabla L(\theta) = \frac{(w^\theta_T, 1)}{(p^\theta_T, 1)}$$

with $(p^\theta_t \, : \, 0 \le t \le T)$ and $(w^\theta_t \, : \, 0 \le t \le T)$ given respectively by (26) and – see (22)

$$dw^\theta_t = \mathcal{L}^*_\theta w^\theta_t \, dt + h^*_\theta w^\theta_t r^{-1} \, dY_t + [\nabla h_\theta]^* p^\theta_t r^{-1} \, dY_t + \mathcal{J}^*_\theta p^\theta_t \, dt \, , \qquad w^\theta_0 = \nabla p^\theta_0 \, , \tag{27}$$

where $\mathcal{J}_\theta \phi \triangleq [\nabla b_\theta]^* D\phi$.

Remark. Equation (27) is exactly what would be obtained by deriving formally equation (26) with respect to the parameter θ. This result was indeed obtained in [4], relying on the existence of a "robust" (i.e. continuous with respect to observation sample paths) version of Zakaï equation. If θ is a p–dimensional parameter, then the gradient $(w^\theta_t \, : \, 0 \le t \le T)$ is a p–dimensional vector: each component of this vector actually solves a stochastic PDE which is coupled only with $(p^\theta_t \, : \, 0 \le t \le T)$ and with no other component; moreover the coupling occurs only through the "right–hand side" and each of these $(p+1)$ stochastic PDE's has the same dynamics. In other words, one has to solve the same stochastic PDE with $(p+1)$ different "right–hand side". Note that smoothing could provide a more efficient method to deal with such a problem.

4.2 The EM algorithm

It follows from (6) and (7) that the auxiliary function $Q(\theta, \theta')$ belongs to the class of conditional expectations considered in Section 3. The approach based on filtering (Theorem 3.2) gives

$$Q(\theta, \theta') = \frac{(w^{\theta\theta'}_T, 1)}{(p^{\theta'}_T, 1)}$$

with $(p_t^{\theta'} : 0 \le t \le T)$ and $(w_t^{\theta\theta'} : 0 \le t \le T)$ given respectively by (26) and – see (22)

$$\begin{aligned} dw_t^{\theta\theta'} &= \mathcal{L}_{\theta'}^* w_t^{\theta\theta'}\, dt + h_{\theta'}^* w_t^{\theta\theta'} r^{-1}\, dY_t + [h_\theta - h_{\theta'}]^* p_t^{\theta'} r^{-1}\, dY_t + \mathcal{J}_{\theta\theta'}^* p_t^{\theta'}\, dt \\ &\quad -\tfrac{1}{2}\left([b_\theta - b_{\theta'}]^* a^{-1}[b_\theta - b_{\theta'}] + [h_\theta - h_{\theta'}]^* r^{-1}[h_\theta - h_{\theta'}]\right) p_t^{\theta'}\, dt\ , \\ w_0^{\theta\theta'} &= p_0^{\theta'} \log \frac{p_0^\theta}{p_0^{\theta'}}\ , \end{aligned}$$

where $\mathcal{J}_{\theta\theta'}\phi \triangleq [b_\theta - b_{\theta'}]^* D\phi$.

On the other hand, smoothing (Theorem 3.5) gives

$$\begin{aligned} Q(\theta,\theta') &= (\rho_0^{\theta'}, \log \frac{p_0^\theta}{p_0^{\theta'}}) + \int_0^T (\rho_s^{\theta'}, [b_\theta - b_{\theta'}]^* D\left(\log \frac{\rho_s^{\theta'}}{\pi_s^{\theta'}}\right))\, ds \\ &\quad -\tfrac{1}{2}\int_0^T (\rho_s^{\theta'}, [b_\theta - b_{\theta'}]^* a^{-1}[b_\theta - b_{\theta'}] + [h_\theta - h_{\theta'}]^* r^{-1}[h_\theta - h_{\theta'}])\, ds + A'\ , \end{aligned} \tag{28}$$

$$A' = \begin{cases} \displaystyle\int_0^T (\rho_s^{\theta'}, [h_\theta - h_{\theta'}]^*) r^{-1}\, dY_s - \int_0^T (\rho_s^{\theta'}, [h_\theta - h_{\theta'}]^*) r^{-1} (\rho_s^{\theta'}, h_{\theta'})\, ds\ , \\ \displaystyle\int_0^T (\rho_s^{\theta'}, [h_\theta - h_{\theta'}]^*) r^{-1} \circ dY_s - \int_0^T (\rho_s^{\theta'}, [h_\theta - h_{\theta'}]^* r^{-1} h_{\theta'})\, ds\ , \end{cases} \tag{29}$$

where $(\pi_t^{\theta'} : 0 \le t \le T)$ and $(\rho_t^{\theta'} : 0 \le t \le T)$ are the normalized density of filtering and smoothing, computed from the unique solution $(p_t^{\theta'} : 0 \le t \le T)$ and $(v_t^{\theta'} : 0 \le t \le T)$ of (26) and – see (19)

$$dv_t^{\theta'} + \mathcal{L}_{\theta'} v_t^{\theta'}\, dt + h_{\theta'}^* v_t^{\theta'} r^{-1}\, dY_t = 0\ , \qquad v_T^{\theta'} \equiv 1\ , \tag{30}$$

respectively. Moreover, the non–adapted stochastic integrals in (29) are respectively a Skorokhod integral and a generalized Stratonovitch integral [6].

Remark. It is now possible to give a more precise meaning to the (E–step) and (M–step) of the algorithm. Indeed, θ' being fixed

3. (E–step) compute the normalized smoothing density $(\rho_t^{\theta'} : 0 \le t \le T)$ – this requires in particular to compute the normalized filtering density $(\pi_t^{\theta'} : 0 \le t \le T)$,
4. (M–step) maximize $Q(\cdot,\theta')$ – where for each $\theta \in \Theta$ the computation of $Q(\theta,\theta')$ requires according to (28) *(i)* at each time t, to integrate some functions depending on (θ,θ') against the normalized smoothing density $\rho_t^{\theta'}$, and *(ii)* to integrate the resulting processes over the interval $[0,T]$.

Remark. A partial answer can be given to the question [M] raised in the Introduction. Indeed

- the differentiability of $\theta \mapsto Q(\theta,\theta')$ relies in an obvious way on the existence of derivatives with respect to θ of $p_0^\theta(\cdot)$, $b_\theta(\cdot)$ and $h_\theta(\cdot)$,
- computing the corresponding derivatives, and maximizing $\theta \mapsto Q(\theta,\theta')$ will not involve the computation of any other infinite–dimensional conditional density.

Moreover, as was pointed out in [2], there are particular cases in which the M–step can be dealt with explicitely. This includes the case where

- $\log p_0^\theta(\cdot)$ depends quadratically on θ,
- $b_\theta(\cdot)$ and $h_\theta(\cdot)$ depend linearly on θ,

since $\theta \mapsto Q(\theta,\theta')$ becomes then a quadratic form.

It follows from (9) and (10) that $\nabla^{1,0}Q(\theta,\theta')$ belongs to the class of conditional expectations considered in Section 3. The approach based on filtering (Theorem 3.2) gives

$$\nabla^{1,0}Q(\theta,\theta') = \frac{(w_T^{\theta\theta'},1)}{(p_T^{\theta'},1)}$$

with $(p_t^{\theta'} : 0 \le t \le T)$ and $(w_t^{\theta\theta'} : 0 \le t \le T)$ given respectively by (26) and – see (22)

$$\begin{aligned} dw_t^{\theta\theta'} &= \mathcal{L}_{\theta'}^* w_t^{\theta\theta'}\, dt + h_{\theta'}^* w_t^{\theta\theta'} r^{-1}\, dY_t + [\nabla h_\theta]^* p_t^{\theta'} r^{-1}\, dY_t + \mathcal{J}_\theta^* p_t^{\theta'}\, dt \\ &\quad - \Big([\nabla b_\theta]^* a^{-1}[b_\theta - b_{\theta'}] + [\nabla h_\theta]^* r^{-1}[h_\theta - h_{\theta'}]\Big)\, p_t^{\theta'}\, dt \\ w_0^{\theta\theta'} &= \frac{p_0^{\theta'}}{p_0^\theta} \nabla p_0^\theta\ , \end{aligned}$$

where $\mathcal{J}_\theta \phi \triangleq [\nabla b_\theta]^* D\phi$.

Remark. Comparing with (27), one can check once again that

$$\nabla^{1,0}Q(\theta,\theta')\,|_{\theta=\theta'} = \nabla L(\theta')\ ,$$

as expected.

As for the smoothing approach, one can use again the results of Section 3. Alternatively, one can directly differentiate with respect to θ the expression (28) for $Q(\theta,\theta')$, thus illustrating the point [M]. Indeed

$$\begin{aligned} \nabla^{1,0}(\theta,\theta') &= \Big(\rho_0^{\theta'}, \frac{\nabla p_0^\theta}{p_0^\theta}\Big) + \int_0^T \Big(\rho_s^{\theta'}, [\nabla b_\theta]^* D\Big(\log\frac{\rho_s^{\theta'}}{\pi_s^{\theta'}}\Big)\Big)\, ds \\ &\quad - \int_0^T (\rho_s^{\theta'}, [\nabla b_\theta]^* a^{-1}[b_\theta - b_{\theta'}] + [\nabla h_\theta]^* r^{-1}[h_\theta - h_{\theta'}])\, ds + A'\ , \\ A' &= \begin{cases} \displaystyle\int_0^T (\rho_s^{\theta'}, [\nabla h_\theta]^*) r^{-1}\, dY_s - \int_0^T (\rho_s^{\theta'}, [\nabla h_\theta]^*) r^{-1} (\rho_s^{\theta'}, h_{\theta'})\, ds\ , \\ \displaystyle\int_0^T (\rho_s^{\theta'}, [\nabla h_\theta]^*) r^{-1} \circ dY_s - \int_0^T (\rho_s^{\theta'}, [\nabla h_\theta]^*) r^{-1} h_{\theta'})\, ds\ , \end{cases} \end{aligned}$$

where the non–adapted stochastic integrals are respectively a Skorokhod integral and a generalized Stratonovitch integral [6].

5 Time–discretization, and relation with MLE of parameters in partially observed Markov chains

Before turning to the presentation of the numerical results, it is worth describing the approach that has been adopted to actually compute the expressions obtained for $L(\theta)$, $\nabla L(\theta)$ and $Q(\theta, \theta')$. From the results of the previous section, this should reduce in some sense to discretizing stochastic PDE's (26), (27) and (30).

However, instead of discretizing separately these stochastic PDE's and e.g. just plugging the resulting approximations into a discretized version of (28), a global approximation of the original continuous–time problem by a discrete–time problem will be presented. In particular

- the approximation $\overline{L}(\theta)$ to the log–likelihood function $L(\theta)$ of the continuous–time problem, will be interpreted as the log–likelihood function of the discrete–time problem,
- the approximation $\overline{Q}(\theta, \theta')$ to the auxiliary function $Q(\theta, \theta')$ of the continuous–time problem will be such that the fundamental relation (2) will hold for the discrete–time problem, i.e. $\overline{L}(\theta) - \overline{L}(\theta') \geq \overline{Q}(\theta, \theta')$.

Consider indeed the following discrete–time statistical model. Let first $(t_n : 0 \leq n \leq N)$ be a uniform partition of the interval $[0, T]$ with time–step Δt. Suppose that on a measurable space $(\Omega, \mathcal{F})$ are given

- a family $(\overline{P}_\theta : \theta \in \Theta)$ of probability measures,
- a discrete–time stochastic process $(\overline{X}_n : 0 \leq n \leq N)$ taking values in $\mathbf{R}^m$,
- a stochastic process $(Y_t : t \geq 0)$ taking values in $\mathbf{R}^d$,

such that under $\overline{P}_\theta$, $(\overline{X}_n : 0 \leq n \leq N)$ is a Markov chain with transition probabilities kernel

$$\Pi_\theta \triangleq (I - \Delta t \mathcal{L}_\theta)^{-1} \tag{31}$$

and initial density p_0^θ, and this Markov chain is observed in continuous–time through

$$dY_t = h_\theta(\overline{X}_n)\, dt + d\overline{W}_t \,, \qquad t_n \leq t < t_{n+1} \,,$$

where $(\overline{W}_t : 0 \leq t \leq T)$ is a Wiener process with matrix covariance r, independent of the Markov chain $(\overline{X}_n : 0 \leq n \leq N)$.

Remark. Equivalently, one can consider that the Markov chain is observed through the discrete-time measurements

$$y_n \triangleq \frac{\Delta Y_n}{\Delta t} = h_\theta(\overline{X}_n) + \overline{w}_n \qquad (\Delta Y_n \triangleq Y_{t_{n+1}} - Y_{t_n}) \,,$$

where $(\overline{w}_n : 0 \leq n \leq N)$ is a Gaussian white noise sequence with matrix covariance $r\Delta t^{-1}$, independent of the Markov chain $(\overline{X}_n : 0 \leq n \leq N)$.

First, it follows from hypotheses $(H_1 - H_2)$ that $\forall x \in \mathbf{R}^m$, $(\Pi_\theta(x,\cdot) : \theta \in \Theta)$ are mutually absolutely continuous probability measures on $\mathbf{R}^m$. Define then

$$f_{\theta,\theta'}(x,y) \triangleq \frac{\Pi_\theta(x,dy)}{\Pi_{\theta'}(x,dy)} ,$$

as the corresponding Radon–Nikodym derivative. Define next

$$\Psi_n^\theta(x) \triangleq \exp\left\{h_\theta^*(x) r^{-1} \Delta Y_n - \tfrac{1}{2} h_\theta^*(x) r^{-1} h_\theta(x) \Delta t\right\} .$$

Then $(\overline{P}_\theta : \theta \in \Theta)$ are mutually absolutely continuous probability measures on $(\Omega, \mathcal{F})$ with Radon–Nikodym derivative

$$\overline{\Lambda}_{\theta,\theta'} \triangleq \frac{d\overline{P}_\theta}{d\overline{P}_{\theta'}} = \frac{p_0^\theta}{p_0^{\theta'}}(\overline{X}_0) \prod_{i=0}^{N-1} f_{\theta,\theta'}(\overline{X}_i, \overline{X}_{i+1}) \prod_{i=0}^{N-1} \frac{\Psi_i^\theta}{\Psi_i^{\theta'}}(\overline{X}_i) .$$

Consider also the probability measure $\overline{P}_\theta^\dagger$ defined by

$$\overline{Z}^\theta \triangleq \frac{d\overline{P}_\theta}{d\overline{P}_\theta^\dagger} = \prod_{i=0}^{N-1} \Psi_i^\theta(\overline{X}_i) ,$$

so that under $\overline{P}_\theta^\dagger$, $(Y_t : 0 \le t \le T)$ is a Wiener process independent of the Markov chain $(\overline{X}_n : 0 \le n \le N)$.

Let again $(\mathcal{Y}_t : 0 \le t \le T)$ denote the observation filtration. It turns out that the log-likelihood function for the estimation of the parameter θ is now defined by

$$\overline{L}(\theta) = \log \overline{\mathbf{E}}_\theta^\dagger(\overline{Z}^\theta \mid \mathcal{Y}_T) , \tag{32}$$

whereas the auxiliary function is defined by

$$\overline{Q}(\theta,\theta') = \overline{\mathbf{E}}_{\theta'}(\log \overline{\Lambda}_{\theta,\theta'} \mid \mathcal{Y}_T) = \frac{\overline{\mathbf{E}}_{\theta'}^\dagger(\log \overline{\Lambda}_{\theta,\theta'} \overline{Z}^{\theta'} \mid \mathcal{Y}_T)}{\overline{\mathbf{E}}_{\theta'}^\dagger(\overline{Z}^{\theta'} \mid \mathcal{Y}_T)} . \tag{33}$$

5.1 Direct maximization of the likelihood function

The idea is to find an equation for $(\overline{p}_n^\theta : 0 \le n \le N)$ defined by

$$(\overline{p}_n^\theta, \phi) \triangleq \overline{\mathbf{E}}_\theta^\dagger(\phi(\overline{X}_n) \overline{Z}_n^\theta \mid \mathcal{Y}_{t_n}) ,$$

where

$$\overline{Z}_n^\theta \triangleq \prod_{i=0}^{n-1} \Psi_i^\theta(\overline{X}_i) .$$

By definition

$$\begin{aligned}
(\overline{p}_{n+1}^\theta, \phi) &= \overline{\mathbf{E}}_\theta^\dagger(\phi(\overline{X}_{n+1}) \overline{Z}_{n+1}^\theta \mid \mathcal{Y}_{t_{n+1}}) \\
&= \overline{\mathbf{E}}_\theta^\dagger(\phi(\overline{X}_{n+1}) \Psi_n^\theta(\overline{X}_n) \overline{Z}_n^\theta \mid \mathcal{Y}_{t_{n+1}}) \\
&= \overline{\mathbf{E}}_\theta^\dagger(\Psi_n^\theta(\overline{X}_n^\theta)\, [\Pi_\theta \phi](\overline{X}_n)\, \overline{Z}_n^\theta \mid \mathcal{Y}_{t_{n+1}}) \\
&= (\overline{p}_n^\theta, \Psi_n^\theta(\Pi_\theta \phi)) ,
\end{aligned}$$

which results in the following equation

$$\overline{p}^{\theta}_{n+1} = \Pi^{*}_{\theta}(\Psi^{\theta}_{n}\overline{p}^{\theta}_{n}) , \qquad \overline{p}^{\theta}_{0} = p^{\theta}_{0} . \tag{34}$$

Using expression (31) for the transition probabilities kernel gives the following discretization scheme of Zakaï equation (26), which combines a Trotter–like product formula and a Euler implicit scheme

$$(I - \Delta t \mathcal{L}^{*}_{\theta})\, \overline{p}^{\theta}_{n+1} = \Psi^{\theta}_{n}\, \overline{p}^{\theta}_{n} , \qquad \overline{p}^{\theta}_{0} = p^{\theta}_{0} . \tag{35}$$

It follows from (32) that the log–likelihood function $L(\theta)$ is therefore approximated by

$$\overline{L}(\theta) = \log(\overline{p}^{\theta}_{N}, 1) . \tag{36}$$

To approximate the gradient $\nabla L(\theta)$, one could either

- directly discretize equation (27),
- derive the exact expression for the gradient of the approximated log–likelihood function $\overline{L}(\theta)$.

The second method is prefered, and gives

$$\nabla \overline{L}(\theta) = \frac{(\overline{w}^{\theta}_{N}, 1)}{(\overline{p}^{\theta}_{N}, 1)} ,$$

with – deriving equation (35) with respect to the parameter θ

$$(I - \Delta t \mathcal{L}^{*}_{\theta})\, \overline{w}^{\theta}_{n+1} = \Psi^{\theta}_{n}\overline{w}^{\theta}_{n} + \Delta t [\nabla \mathcal{L}^{*}_{\theta}]\, \overline{p}^{\theta}_{n+1} + [\nabla \Psi^{\theta}_{n}]\, \overline{p}^{\theta}_{n} , \qquad \overline{w}^{\theta}_{0} = \nabla p^{\theta}_{0} .$$

Remark. (normalization) To avoid numerical overflow one should rather solve, instead of (35), the normalized equations

$$\left.\begin{aligned} \overline{\pi}^{\theta}_{n+\frac{1}{2}} &= \Psi^{\theta}_{n}\, \overline{\pi}^{\theta}_{n} / l^{\theta}_{n+1} \\ (I - \Delta t \mathcal{L}^{*}_{\theta})\, \overline{\pi}^{\theta}_{n+1} &= \overline{\pi}^{\theta}_{n+\frac{1}{2}} \end{aligned}\right\} \qquad \overline{\pi}^{\theta}_{0} = p^{\theta}_{0} ,$$

where $l^{\theta}_{n+1} \triangleq (\overline{\pi}^{\theta}_{n}, \Psi^{\theta}_{n})$. It is easily seen that $\overline{p}^{\theta}_{n} = \gamma^{\theta}_{n}\overline{\pi}^{\theta}_{n}$ with $\gamma^{\theta}_{n} \triangleq l^{\theta}_{n} \cdot l^{\theta}_{n-1} \cdots l^{\theta}_{1}$ and $(\overline{p}^{\theta}_{n}, 1) = \gamma^{\theta}_{n}$ so that

$$\overline{L}(\theta) = \log \gamma^{\theta}_{N} = \sum_{i=1}^{N-1} \log (\overline{\pi}^{\theta}_{i}, \Psi^{\theta}_{i}) .$$

In the same way, defining $\overline{\overline{w}}^{\theta}_{n}$ by the relation $\overline{w}^{\theta}_{n} = \gamma^{\theta}_{n}\overline{\overline{w}}^{\theta}_{n}$ gives

$$\left.\begin{aligned} (I - \Delta t \mathcal{L}^{*}_{\theta})\, \overline{\overline{w}}^{\theta}_{n+\frac{1}{2}} &= \Psi^{\theta}_{n}\, \overline{\overline{w}}^{\theta}_{n} + \Delta t [\nabla \mathcal{L}^{*}_{\theta}]\, \overline{\pi}^{\theta}_{n+\frac{1}{2}} + [\nabla \Psi^{\theta}_{n}]\, \overline{\pi}^{\theta}_{n} \\ \overline{\overline{w}}^{\theta}_{n+1} &= \overline{\overline{w}}^{\theta}_{n+\frac{1}{2}} \cdot (l^{\theta}_{n+1})^{-1} \end{aligned}\right\} \qquad \overline{\overline{w}}^{\theta}_{0} = \nabla p^{\theta}_{0} .$$

Note that, although $\overline{w}^{\theta}_{n}$ is the gradient of $\overline{p}^{\theta}_{n}$, $\overline{\overline{w}}^{\theta}_{n}$ is *not* the gradient of $\overline{\pi}^{\theta}_{n}$. Actually $\overline{\overline{w}}^{\theta}_{n} = \overline{w}^{\theta}_{n} / (\overline{p}^{\theta}_{n}, 1)$ so that

$$\nabla \overline{L}(\theta) = (\overline{\overline{w}}^{\theta}_{N}, 1) .$$

5.2 The EM algorithm

Although it is rather straightforward, in the discrete–time case, to obtain the expression of the auxiliary function $\overline{Q}(\cdot,\cdot)$ in terms of nonlinear smoothing, it is nevertheless worth presenting a derivation that follows the same lines as in the continuous–time case. Indeed, there are two different methods – one based on nonlinear filtering, the other on nonlinear smoothing – for the computation of (33).

- *Filtering*

Define

$$\overline{\lambda}_n^{\theta,\theta'} = \log \frac{p_0^\theta}{p_0^{\theta'}}(\overline{X}_0) + \sum_{i=0}^{n-1} \log f_{\theta,\theta'}(\overline{X}_i, \overline{X}_{i+1}) + \sum_{i=0}^{n-1} \log \frac{\Psi_i^\theta}{\Psi_i^{\theta'}}(\overline{X}_i) .$$

The idea again is to find an equation for $(\overline{w}_n^{\theta,\theta'} \, : \, 0 \leq n \leq N)$ defined by

$$(\overline{w}_n^{\theta,\theta'}, \phi) \triangleq \overline{\mathbf{E}}_{\theta'}^\dagger(\phi(\overline{X}_n)\overline{\lambda}_n^{\theta,\theta'}\overline{Z}_n^{\theta'} \mid \mathcal{Y}_{t_n}) .$$

First

$$\overline{w}_0^{\theta,\theta'} = p_0^{\theta'} \log \frac{p_0^\theta}{p_0^{\theta'}} .$$

Next, by definition

$$\begin{aligned}
(\overline{w}_{n+1}^{\theta,\theta'}, \phi) &= \overline{\mathbf{E}}_{\theta'}^\dagger(\phi(\overline{X}_{n+1})\overline{\lambda}_{n+1}^{\theta,\theta'}\overline{Z}_{n+1}^{\theta'} \mid \mathcal{Y}_{t_{n+1}}) \\
&= \overline{\mathbf{E}}_{\theta'}^\dagger(\phi(\overline{X}_{n+1})\Psi_n^{\theta'}(\overline{X}_n)[\overline{\lambda}_n^{\theta,\theta'} + \log f_{\theta,\theta'}(\overline{X}_n, \overline{X}_{n+1}) + \log \frac{\Psi_n^\theta}{\Psi_n^{\theta'}}(\overline{X}_n)]\overline{Z}_n^{\theta'} \mid \mathcal{Y}_{t_{n+1}}) \\
&= \overline{\mathbf{E}}_{\theta'}^\dagger(\Psi_n^{\theta'}(\overline{X}_n)\,[\Pi_{\theta'}\phi](\overline{X}_n)\,\overline{\lambda}_n^{\theta,\theta'}\,\overline{Z}_n^{\theta'} \mid \mathcal{Y}_{t_{n+1}}) + \overline{\mathbf{E}}_{\theta'}^\dagger(\Psi_n^{\theta'}(\overline{X}_n)\,[\kappa_{\theta,\theta'}\phi](\overline{X}_n)\,\overline{Z}_n^{\theta'} \mid \mathcal{Y}_{t_{n+1}}) \\
&\quad + \overline{\mathbf{E}}_{\theta'}^\dagger(\Psi_n^{\theta'}(\overline{X}_n) \log \frac{\Psi_n^\theta}{\Psi_n^{\theta'}}(\overline{X}_n)\,[\Pi_{\theta'}\phi](\overline{X}_n)\,\overline{Z}_n^{\theta'} \mid \mathcal{Y}_{t_{n+1}}) \\
&= (\overline{w}_n^{\theta,\theta'}, \Psi_n^{\theta'}(\Pi_{\theta'}\phi)) + (\overline{p}_n^{\theta'}, \Psi_n^{\theta'}(\kappa_{\theta,\theta'}\phi)) + (\overline{p}_n^{\theta'}, \Psi_n^{\theta'} \log \frac{\Psi_n^\theta}{\Psi_n^{\theta'}}(\Pi_{\theta'}\phi)) ,
\end{aligned}$$

where the operator $\kappa_{\theta,\theta'}$ is defined by

$$(\kappa_{\theta,\theta'}\phi)(x) \triangleq \int \phi(y) \log f_{\theta,\theta'}(x,y)\, \Pi_{\theta'}(x,dy) . \tag{37}$$

Therefore, the resulting equation is

$$\overline{w}_{n+1}^{\theta,\theta'} = \Pi_{\theta'}^*(\Psi_n^{\theta'}\overline{w}_n^{\theta,\theta'}) + \kappa_{\theta,\theta'}^*(\Psi_n^{\theta'}\overline{p}_n^{\theta'}) + \Pi_{\theta'}^*(\Psi_n^{\theta'} \log \frac{\Psi_n^\theta}{\Psi_n^{\theta'}}\overline{p}_n^{\theta'}) , \qquad \overline{w}_0^{\theta,\theta'} = p_0^{\theta'} \log \frac{p_0^\theta}{p_0^{\theta'}} .$$

It follows from (33) that the auxiliary function $Q(\theta,\theta')$ is approximated by

$$\overline{Q}(\theta,\theta') = \frac{(\overline{w}_N^{\theta,\theta'}, 1)}{(\overline{p}_N^{\theta'}, 1)} . \tag{38}$$

- *Smoothing*

Introduce the backward equation – dual to (34)

$$\overline{v}_n^{\theta'} = \Psi_n^{\theta'}(\Pi_{\theta'}\overline{v}_{n+1}^{\theta'}) , \qquad \overline{v}_N^{\theta'} \equiv 1 . \tag{39}$$

Then

$$\begin{aligned}(\overline{w}_{n+1}^{\theta,\theta'}, \overline{v}_{n+1}^{\theta'}) &= (\Pi_{\theta'}^*(\Psi_n^{\theta'}\overline{w}_n^{\theta,\theta'}), \overline{v}_{n+1}^{\theta'}) + (\kappa_{\theta,\theta'}^*(\Psi_n^{\theta'}\overline{p}_n^{\theta'}), \overline{v}_{n+1}^{\theta'}) + (\Pi_{\theta'}^*(\Psi_n^{\theta'} \log \frac{\Psi_n^{\theta}}{\Psi_n^{\theta'}}\overline{p}_n^{\theta'}), \overline{v}_{n+1}^{\theta'}) \\ &= (\overline{w}_n^{\theta,\theta'}, \overline{v}_n^{\theta'}) + (\Psi_n^{\theta'}\overline{p}_n^{\theta'}, [\kappa_{\theta,\theta'}\overline{v}_{n+1}^{\theta'}]) + (\overline{p}_n^{\theta'}\overline{v}_n^{\theta'}, \log \frac{\Psi_n^{\theta}}{\Psi_n^{\theta'}}) .\end{aligned}$$

Introducing the unnormalized smoothing density $\overline{q}_i^{\theta'} = \overline{p}_i^{\theta'}\overline{v}_i^{\theta'}$, gives

$$\begin{aligned}(\overline{w}_N^{\theta,\theta'}, 1) &= (\overline{w}_0^{\theta,\theta'}, \overline{v}_0^{\theta'}) + \sum_{i=0}^{N-1} [(\overline{w}_{i+1}^{\theta,\theta'}, \overline{v}_{i+1}^{\theta'}) - (\overline{w}_i^{\theta,\theta'}, \overline{v}_i^{\theta'})] \\ &= (\overline{q}_0^{\theta'}, \log \frac{p_0^{\theta}}{p_0^{\theta'}}) + \sum_{i=0}^{N-1} (\Psi_i^{\theta'}\overline{p}_i^{\theta'}, [\kappa_{\theta,\theta'}\overline{v}_{i+1}^{\theta'}]) + \sum_{i=0}^{N-1} (\overline{q}_i^{\theta'}, \log \frac{\Psi_n^{\theta}}{\Psi_n^{\theta'}}) \\ &= (\overline{q}_0^{\theta'}, \log \frac{p_0^{\theta}}{p_0^{\theta'}}) + \sum_{i=0}^{N-1} (\overline{p}_{i+\frac{1}{2}}^{\theta'}, [\kappa_{\theta,\theta'}\overline{v}_{i+1}^{\theta'}]) \\ &\quad + \sum_{i=0}^{N-1} (\overline{q}_i^{\theta'}, [h_\theta - h_{\theta'}]^* r^{-1}(\Delta Y_i - h_{\theta'}\Delta t)) - \tfrac{1}{2} \sum_{i=0}^{N-1} (\overline{q}_i^{\theta'}, [h_\theta - h_{\theta'}]^* r^{-1}[h_\theta - h_{\theta'}])\Delta t ,\end{aligned}$$

where in the last expression $\overline{p}_{i+\frac{1}{2}}^{\theta'} \triangleq \Psi_i^{\theta'}\overline{p}_i^{\theta'}$, and the identity

$$\log \frac{\Psi_i^{\theta}}{\Psi_i^{\theta'}} = [h_\theta - h_{\theta'}]^* r^{-1}(\Delta Y_i - h_{\theta'}\Delta t) - \tfrac{1}{2}[h_\theta - h_{\theta'}]^* r^{-1}[h_\theta - h_{\theta'}]\Delta t$$

has been used.

Remark. (normalization) Here again one should rather solve, instead of (39), the normalized equations

$$\left.\begin{aligned}\overline{\overline{v}}_{n+\frac{1}{2}}^{\theta'} &= \Pi_{\theta'}\overline{\overline{v}}_{n+1}^{\theta'} \\ \overline{\overline{v}}_n^{\theta'} &= \Psi_n^{\theta'}\overline{\overline{v}}_{n+\frac{1}{2}}^{\theta'}/j_n^{\theta'}\end{aligned}\right\} \qquad \overline{\overline{v}}_N^{\theta'} \equiv 1 ,$$

where $j_n^{\theta'}$ is chosen in such a way that $(\overline{\pi}_n^{\theta'}, \overline{\overline{v}}_n^{\theta'}) = 1$, which gives $j_n^{\theta'} = (\overline{\pi}_n^{\theta'}, \Psi_n^{\theta'}\overline{\overline{v}}_{n+\frac{1}{2}}^{\theta'})$. It is then easily seen that $j_n^{\theta'} = l_{n+1}^{\theta'}$, and that $\overline{v}_n^{\theta'} = \delta_n^{\theta'}\overline{\overline{v}}_n^{\theta'}$ with $\delta_n^{\theta'} \triangleq j_n^{\theta'} \cdot j_{n+1}^{\theta'} \cdots j_N^{\theta'}$. Moreover, the normalized smoothing density is given by $\overline{\rho}_n^{\theta'} \triangleq \overline{\pi}_n^{\theta'}\overline{\overline{v}}_n^{\theta'}$.

Remark. In terms of normalized conditional densities

$$\overline{Q}(\theta,\theta') = (\overline{p}_0^{\theta'}, \log \frac{p_0^\theta}{p_0^{\theta'}}) + \sum_{i=0}^{N-1} (\overline{\pi}_{i+\frac{1}{2}}^{\theta'}, [\kappa_{\theta,\theta'} \overline{v}_{i+1}^{\theta'}])$$
$$+ \sum_{i=0}^{N-1} (\overline{\rho}_i^{\theta'}, [h_\theta - h_{\theta'}]^* r^{-1} (\Delta Y_i - h_{\theta'} \Delta t)) - \tfrac{1}{2} \sum_{i=0}^{N-1} (\overline{\rho}_i^{\theta'}, [h_\theta - h_{\theta'}]^* r^{-1} [h_\theta - h_{\theta'}]) \Delta t$$

to be compared with (28), (29).

Remark. It is now possible to give a more precise meaning to the (E–step) and (M–step) of the algorithm. Indeed, θ' being fixed

3. (E–step) compute the normalized smoothing density $(\overline{\rho}_n^{\theta'} : 0 \leq n \leq N)$ – this requires in particular to compute the normalized filtering density $(\overline{\pi}_n^{\theta'} : 0 \leq n \leq N)$,
4. (M–step) maximize $\overline{Q}(\cdot,\theta')$ – where for each $\theta \in \Theta$ the computation of $\overline{Q}(\theta,\theta')$ requires *(i)* at each time n, to integrate some functions depending on (θ,θ') against the normalized smoothing density $\overline{\rho}_n^{\theta'}$, and *(ii)* to sum the resulting discrete–time processes from $n = 0$ to $n = N - 1$.

Remark. With the time–discretization introduced above, the numerical implementation (including discretization with respect to the space variable) of the EM algorithm requires in the M–step, the explicit evaluation of the transition probabilities kernel $\Pi_\theta = (I - \Delta t \mathcal{L}_\theta)^{-1}$. On the other hand, the numerical implementation of the direct maximization algorithm requires only the solution of linear equations with operator $(I - \Delta t \mathcal{L}_\theta^*)$, a much faster task.

Remark. There are some similarity between the discrete–time version of the EM algorithm and the statistical estimation of probabilistic functions of Markov processes. This theory has been introduced in [1], and has found interesting applications in acoustic speech recognition [5]. Indeed, assume that observations are generated according to a hidden Markov model (HMM): to each possible state x of the a non–observed Markov chain defined by its initial probability p_0 and its transition probabilities kernel Π, is associated a probability function $B(x,\cdot)$ which describes the conditional law of the observation given that the chain is in state x. Such a model will be denoted by $\mathcal{M} = (p_0, \Pi, B)$. Then (under the additional assumption that both the Markov chain and the observation sequence take values in finite sets), the maximum likelihood estimation of the parameters of the hidden Markov model $\mathcal{M}$ is achieved by an iterative procedure involving *reestimation formulas* [1,5], which are obtained from the explicit maximization of an auxiliary function $Q(\mathcal{M},\mathcal{M}')$.

Consider now the parametric model described above. It is possible to turn it into a parametric hidden Markov model $\mathcal{M}_\theta = (p_0^\theta, \Pi_\theta, B_\theta)$ with

$$B_\theta(x,y) = (2\pi)^{-\frac{d}{2}} (\det r)^{-\frac{1}{2}} \exp\left\{-\tfrac{1}{2}[h_\theta(x) - y]^* r^{-1} [h_\theta(x) - y] \Delta t\right\} .$$

In particular $B_\theta(x, y_n) \propto \Psi_n^\theta(x)$. Then it is easily seen that the auxiliary function defined in (33) is such that $\overline{Q}(\theta,\theta') = Q(\mathcal{M}_\theta, \mathcal{M}_{\theta'})$. Moreover, equations (34) and (39) – which are known

as Baum's forward and backward equations [1,5] – play a central role in the theory of statistical estimation of hidden Markov models.

6 Numerical example

The continouous–time model is described by

$$dX_t = -\theta_2 X_t\,dt + \theta_3 \frac{X_t}{1+X_t^2}\,dt + a^{1/2}\,dW_t\ , \qquad X_0 \sim \mathcal{N}(\theta_1, \Sigma)\ , \tag{40}$$

$$dY_t = \theta_4 \arctan(\frac{X_t}{\theta_4})\,dt + r^{1/2}\,d\overline{W}_t\ , \tag{41}$$

and the unkown parameter is $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$. The noises covariances in the problem are Σ, a and r, and can be associated with the parameters θ_1, (θ_2, θ_3) and θ_4 respectively.

Although the unknown parameter is actually four–dimensional, results will be presented for the estimation of one component of θ at a time, and the influence of the "associated" noise covariance will be investigated.

For each of the cases presented below, the log–likelihood function has been maximized in order to find the MLE, either using the direct approach or the EM algorithm based on nonlinear smoothing. To achieve the direct maximization, one can rely on existing minimization routines from a scientific library, e.g. `e04jbf` from NAG which uses a quasi–Newton algorithm and does not require the user to provide a routine for the computation of the gradient. On the other hand, the M–step of the EM algorithm can either

- be solved explicitely when applicable, e.g. when the auxiliary function depends quadratically on the parameter to be estimated,
- rely on routines from a scientific library.

Two figures are given for each of the cases considered. On the first figure, the following objects can be found

- *in solid line:* the log–likelihood function $\overline{L}(\cdot)$ vs. the free parameter,
- *in dashed line:* iterations of the quasi–Newton algorithm for the direct maximization of the log–likelihood function $\overline{L}(\cdot)$, i.e. straight lines connecting successive points

  $$A_0, A_1, \cdots, A_n, \cdots\ ,$$

  defined by

  $$A_n \triangleq (\widehat{\theta}_n, \overline{L}(\widehat{\theta}_n))\ .$$

On the second figure, the following objects can be found

- *in solid line:* the log–likelihood function $\overline{L}(\cdot)$ vs. the free parameter,
- *in dotted lines:* the auxiliary functions corresponding to successive estimates, i.e. functions $\overline{L}_n(\cdot) \triangleq \overline{Q}(\cdot, \widehat{\theta}_n) + \overline{L}(\widehat{\theta}_n)$, vs. the free parameter,

- *in dashed lines:* iterations of the EM algorithm, i.e. straight lines connecting successive points $A_0, B_0, A_1, B_1, \cdots, A_n, B_n, \cdots$ defined by

$$A_n \triangleq (\widehat{\theta}_n, \overline{L}(\widehat{\theta}_n)) ,$$
$$B_n \triangleq (\widehat{\theta}_{n+1}, \overline{L}_n(\widehat{\theta}_{n+1})) .$$

Remark. In the example introduced above, although the auxiliary function $Q(\theta, \theta')$ of the continuous–time model depends quadratically on the parameters θ_1, θ_2 and θ_3, the discrete–time approximation $\overline{Q}(\theta, \theta')$ depends quadratically on θ_1 only. This can be seen on the expression of the operator $\kappa_{\theta,\theta'}$ – see (37).

Description of cases study

In all these cases, the "true" value of the parameter – i.e. the value used for simulating sample paths of the observation process – is $(\theta_1^*, \theta_2^*, \theta_3^*, \theta_4^*) = (1.0, 0.25, 5.0, 2.0)$.

The time interval is $[0, T]$ with $T = 10.0$ and time–step $\Delta t = 0.1$. Observation process sample paths are simulated in the following way. First, simple Euler time–discretization scheme (equivalent on this particular example to Milshtein scheme) is used to simulate the signal process (40)

$$x_{n+1} = x_n + [-\theta_2 x_n + \theta_3 \frac{x_n}{1 + x_n^2}]\Delta t + w_n ,$$

with $x_0 \sim \mathcal{N}(\theta_1, \Sigma)$ and $(w_n : 0 \le n \le N)$ a Gaussian white noise sequence with covariance matrix $a\Delta t$. Next, discrete measurements are generated by

$$y_n = \theta_4 \arctan(\frac{x_n}{\theta_4}) + \overline{w}_n ,$$

with $(\overline{w}_n : 0 \le n \le N)$ a Gaussian white noise sequence with matrix covariance $r\Delta t^{-1}$, independent of $(w_n : 0 \le n \le N)$.

These discrete measurements are used to solve equations (34) and (39), and therefore to compute the approximations $\overline{L}(\theta)$ and $\overline{Q}(\theta, \theta')$ defined by (32) and (33) respectively.

- ***Estimation of*** θ_1

Fixed parameters: $(\theta_2, \theta_3, \theta_4) = (\theta_2^*, \theta_3^*, \theta_4^*)$.
Noises variances: $a = 1.0$, $r = 1.0$, and $\Sigma = 1.0$ (Case I – fig. 1 and 2) or $\Sigma = 0.01$ (Case II – fig. 3 and 4).
In Case I the EM algorithm has converged after 11 iterations, whereas in Case II it has not converged after 200 iterations. Therefore, only the 12 first iterations are shown on fig. 4.

- ***Estimation of*** θ_3

Fixed parameters: $(\theta_1, \theta_2, \theta_4) = (\theta_1^*, \theta_2^*, \theta_4^*)$.
Noises variances: $\Sigma = 1.0$, $r = 1.0$, and $a = 1.0$ (Case III – fig. 5 and 6) or $a = 0.01$ (Case IV –

fig. 7 and 8).
In Case III the EM algorithm has converged after 5 iterations, whereas in Case IV it has not converged after 200 iterations. Therefore, only the 12 first iterations are shown on fig. 8.

• *Estimation of θ_4*

Fixed parameters: $(\theta_1, \theta_2, \theta_3) = (\theta_1^*, \theta_2^*, \theta_3^*)$.
Noises variances: $\Sigma = 1.0$, $a = 1.0$, and $r = 1.0$ (Case V – fig. 9 and 10) or $r = 0.01$ (Case VI – fig. 11 and 12).
In Case V the EM algorithm has converged after 9 iterations, whereas in Case VI it has converged after 27 iterations.

The reason why the EM algorithm is so slowly convergent when noise covariances are small – Case II, IV and VI – is that the log–likelihood function is then approximated from below by a set of very sharp auxiliary functions: this situation does not allow to update significantly enough the current estimate at each M–step. Actually, this can be seen directly from (6), (7) – or equivalently from (28), (29). Assume for instance that both $p_0^\theta(\cdot)$ and $b_\theta(\cdot)$ are independent of θ, and that the observation noise covariance r is small. Then every auxiliary function $Q(\cdot, \theta')$ will certainly be very sharp. It should be stressed that in such cases, the slow variation of the estimate should not be interpreted as an indication that the algorithm has already achieved convergence, as one would possibly conclude.

7 Conclusion

The direct maximization of the log–likelihood function has been compared with the EM algorithm, for the MLE of parameters in partially observed diffusion processes. Some formulas given in [2] have been clarified, and it has been shown that smoothing is necesary to make the EM algorithm approach efficient. On the other hand, formulas have been given in terms of filtering stochastic PDE's for the computation of the original log–likelihood function and its gradient.

It has been shown that

[E] the E–step in the EM algorithm is certainly slower than the direct computation of the log–likelihood function, since it involves nonlinear smoothing instead of nonlinear filtering.

[M] the computation of the auxiliary function $Q(\theta, \theta')$ in the M–step of the EM algorithm, θ' being fixed, requires *(i)* at each time t, to integrate some functions depending on (θ, θ') against a normalized smoothing density depending only on θ', and *(ii)* to integrate the resulting processes over the interval $[0, T]$. This gives another evidence that the EM algorithm is more complicated than the direct approach as far as computations are concerned. On the other hand, the maximization of the auxiliary function is generally simple to deal with.

[EM] the EM algorithm converges very slowly whenever some noise covariances associated with the parameters to be estimated are small.

However, the EM algorithm should provide an interesting approach for non–parametric estimation in the context of partially observed diffusion processes, i.e. non–parametric estimation of the initial density, the drift and the observation function. This form of the EM algorithm is used indeed in the context of finite–space Markov chains with finite–state observations (hidden Markov models), and leads to well–known reestimation formulas, which are of practical use e.g. in acoustic speech recognition.

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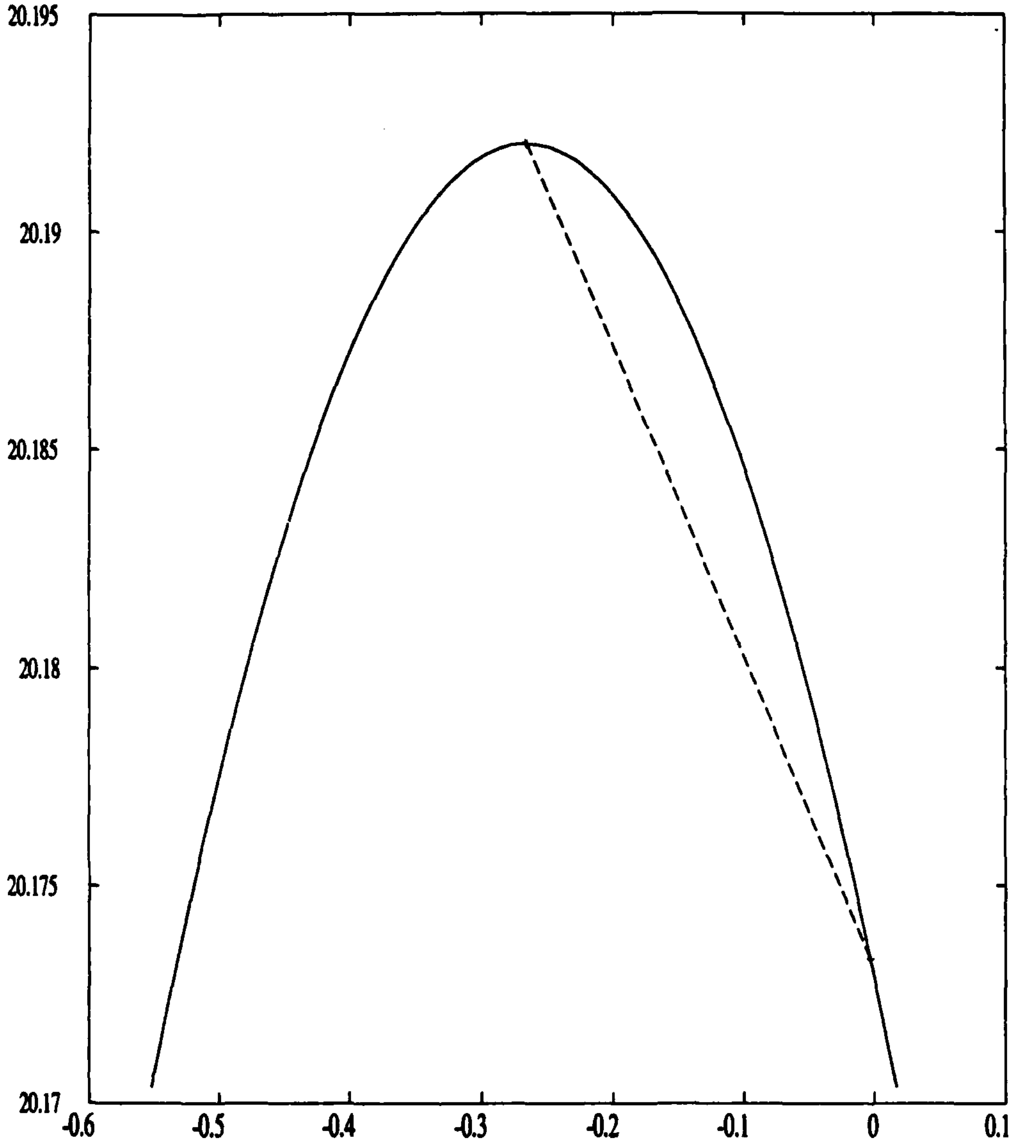

Figure 1: Case I – Direct maximization

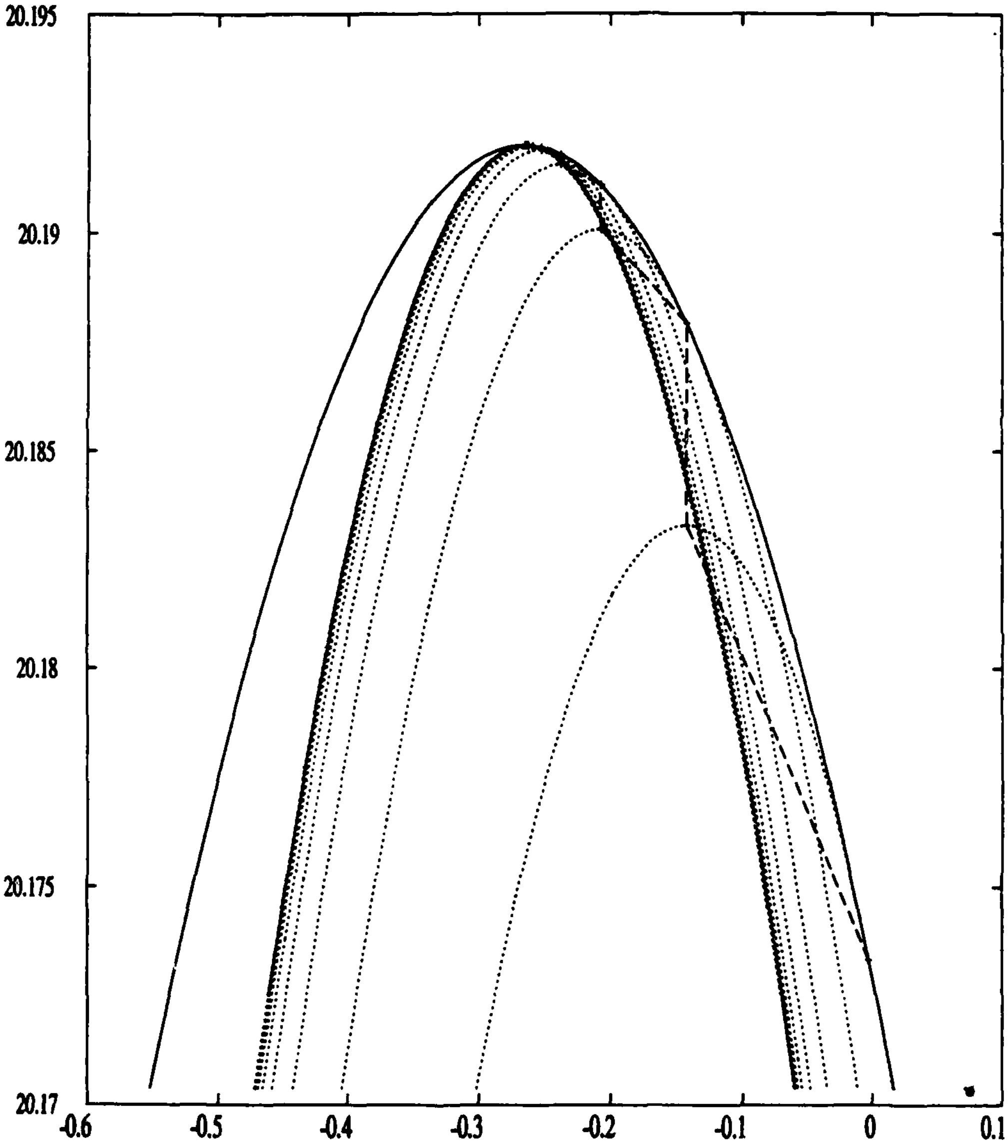

Figure 2: Case I – EM algorithm

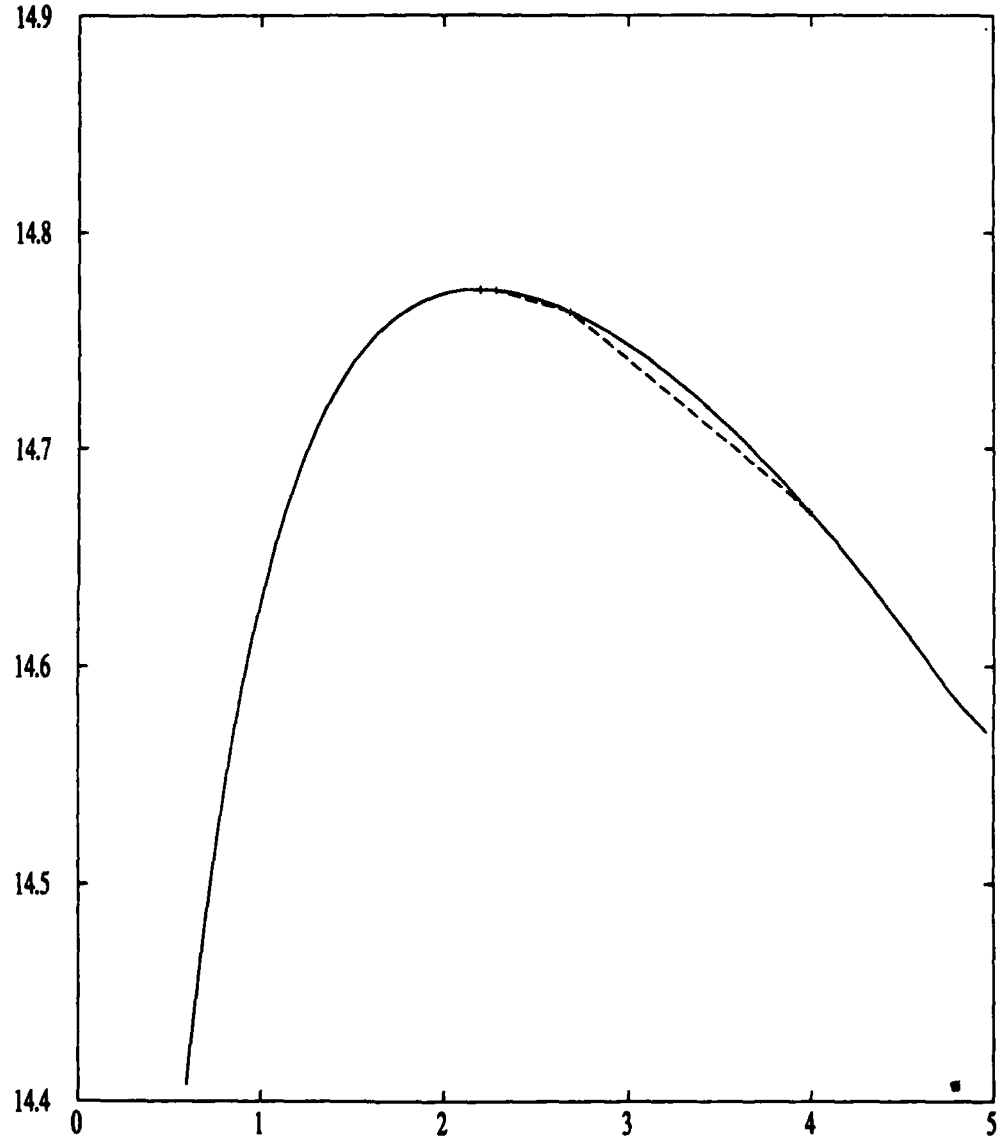

Figure 3: Case II – Direct maximization

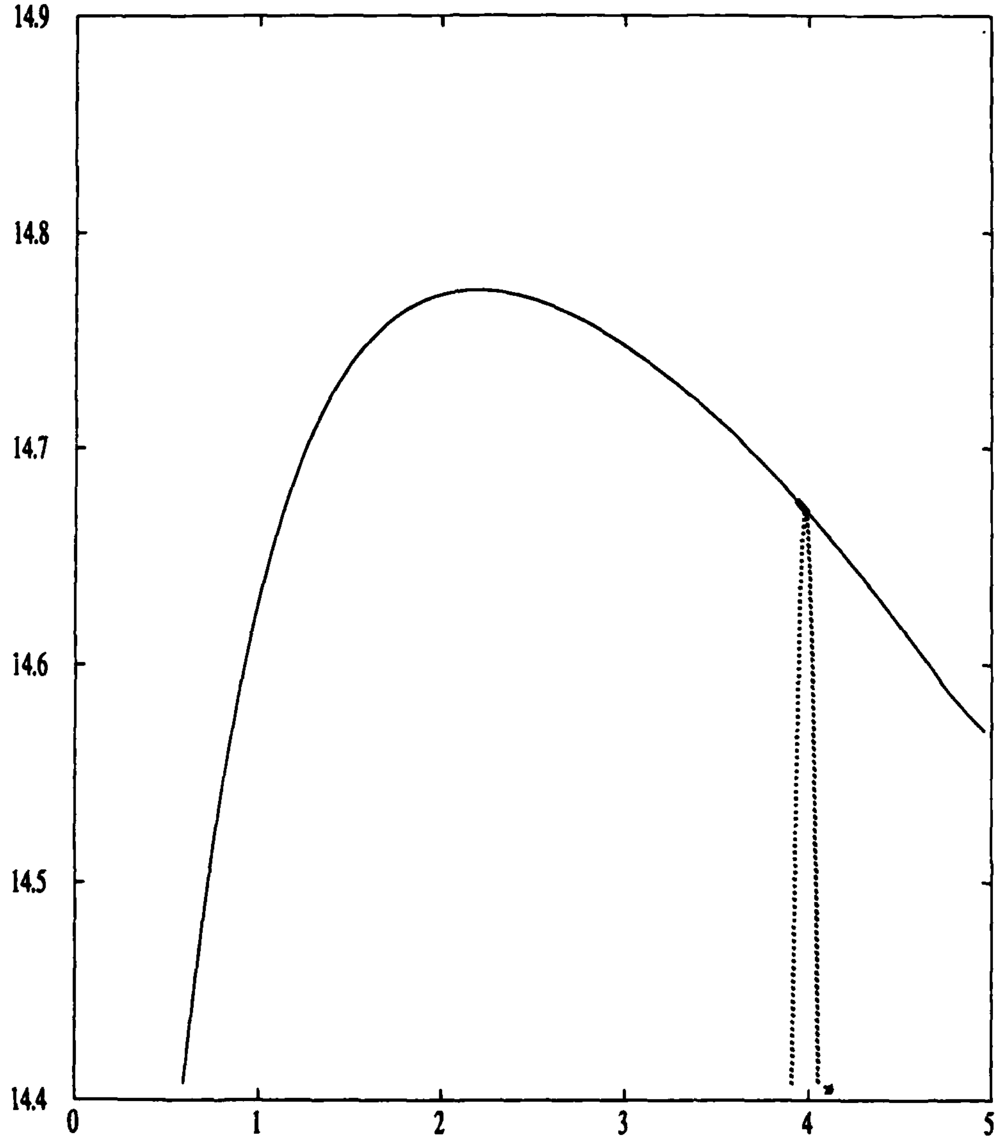

Figure 4: Case II – EM algorithm

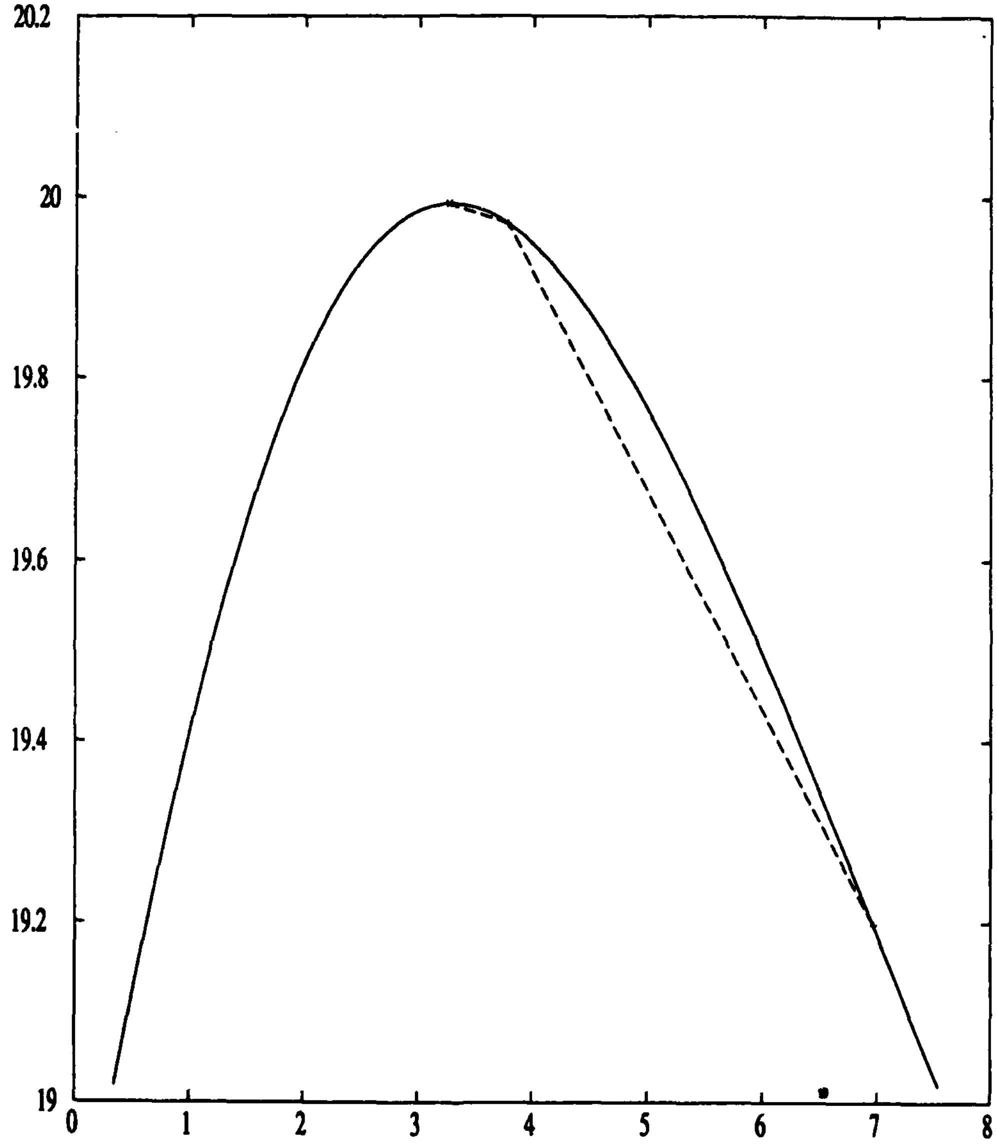

Figure 5: Case III – Direct maximization

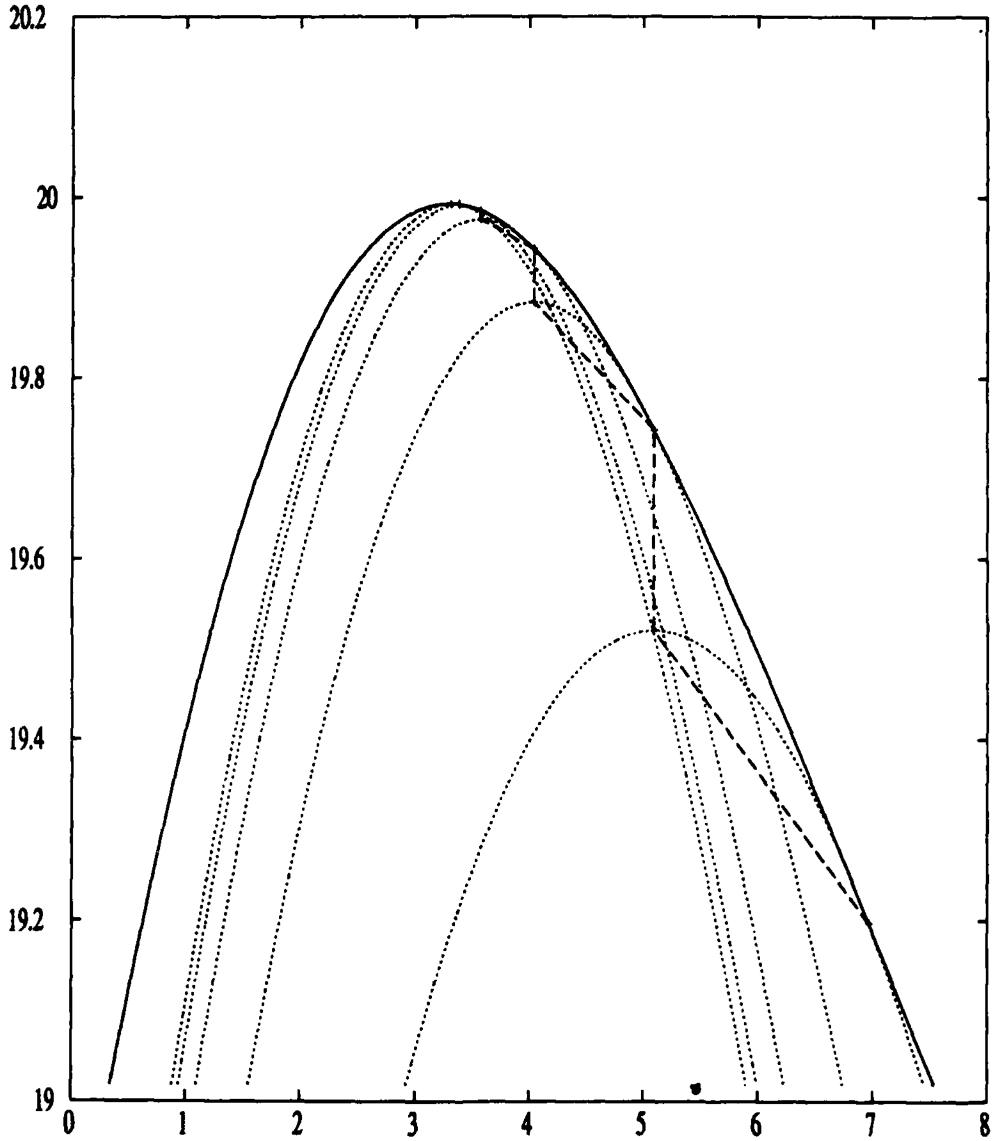

Figure 6: Case III – EM algorithm

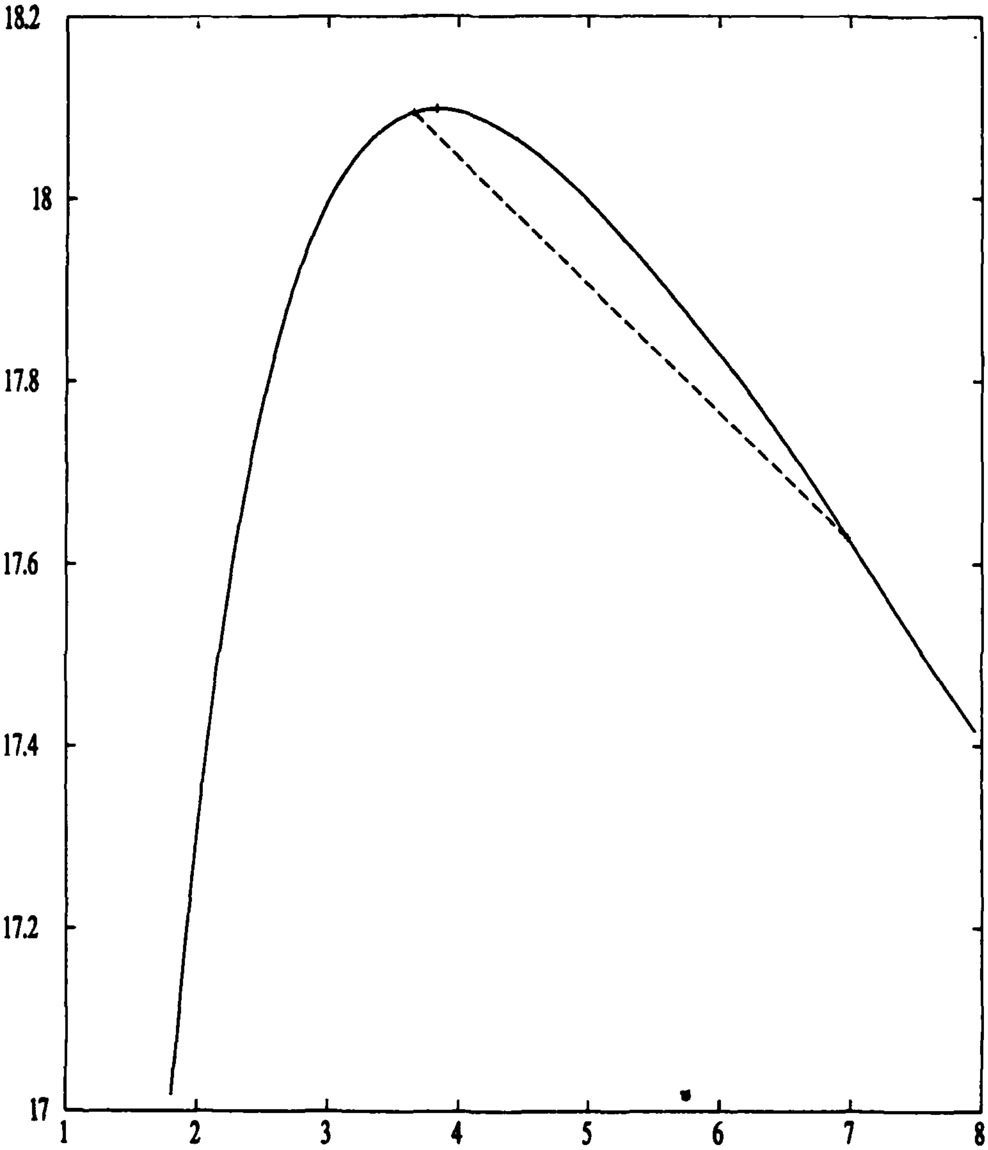

Figure 7: Case IV - Direct maximization

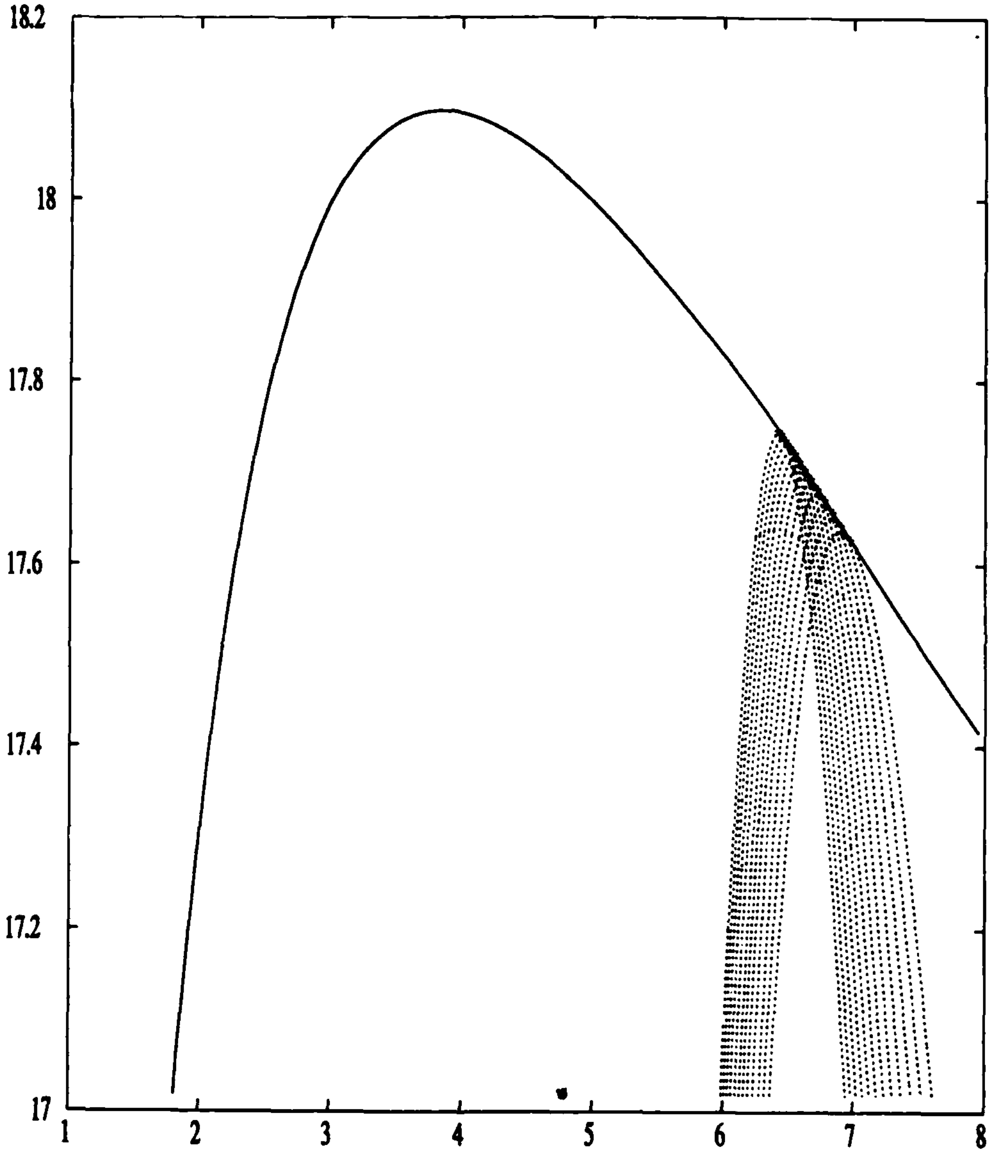

Figure 8: Case IV – EM algorithm

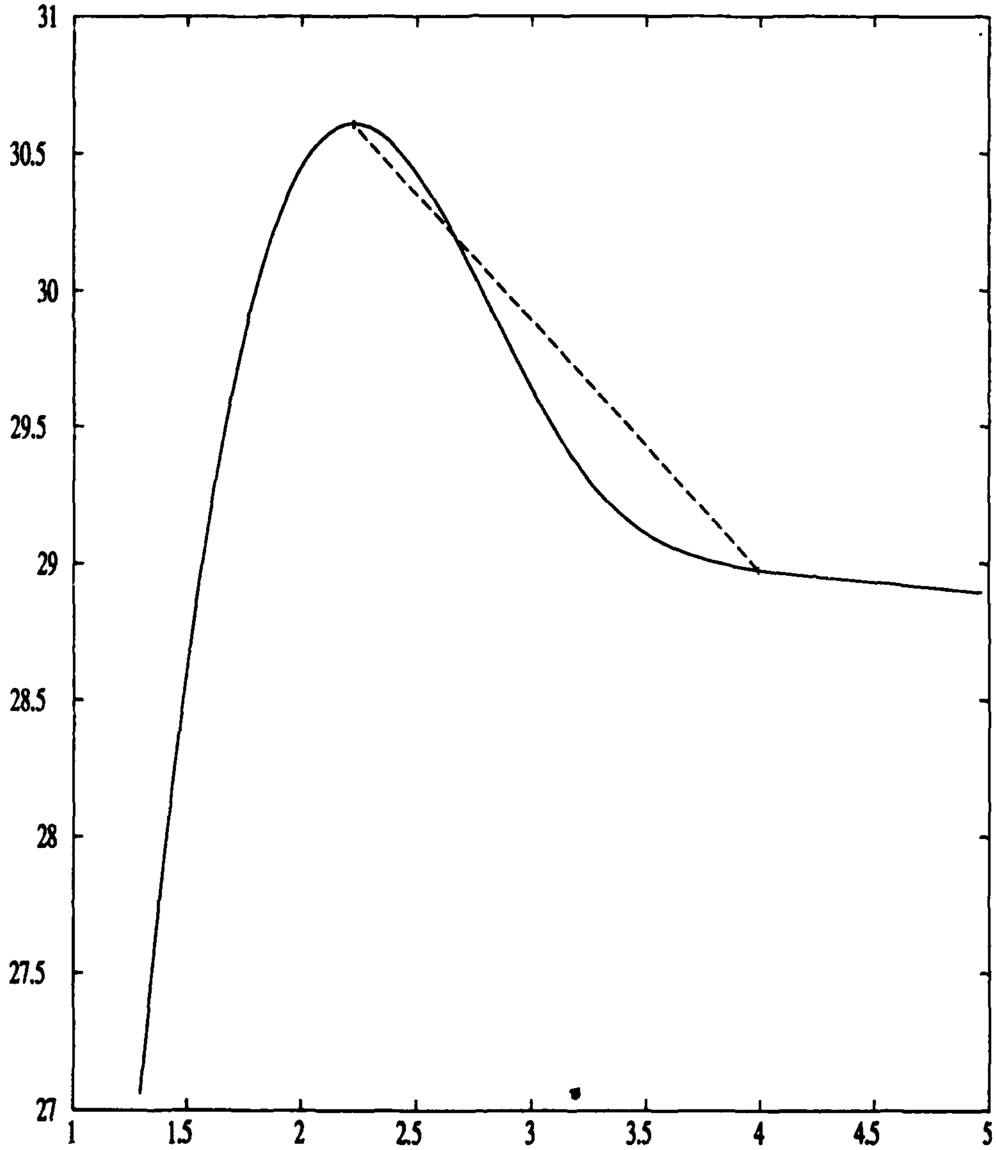

Figure 9: Case V – Direct maximization

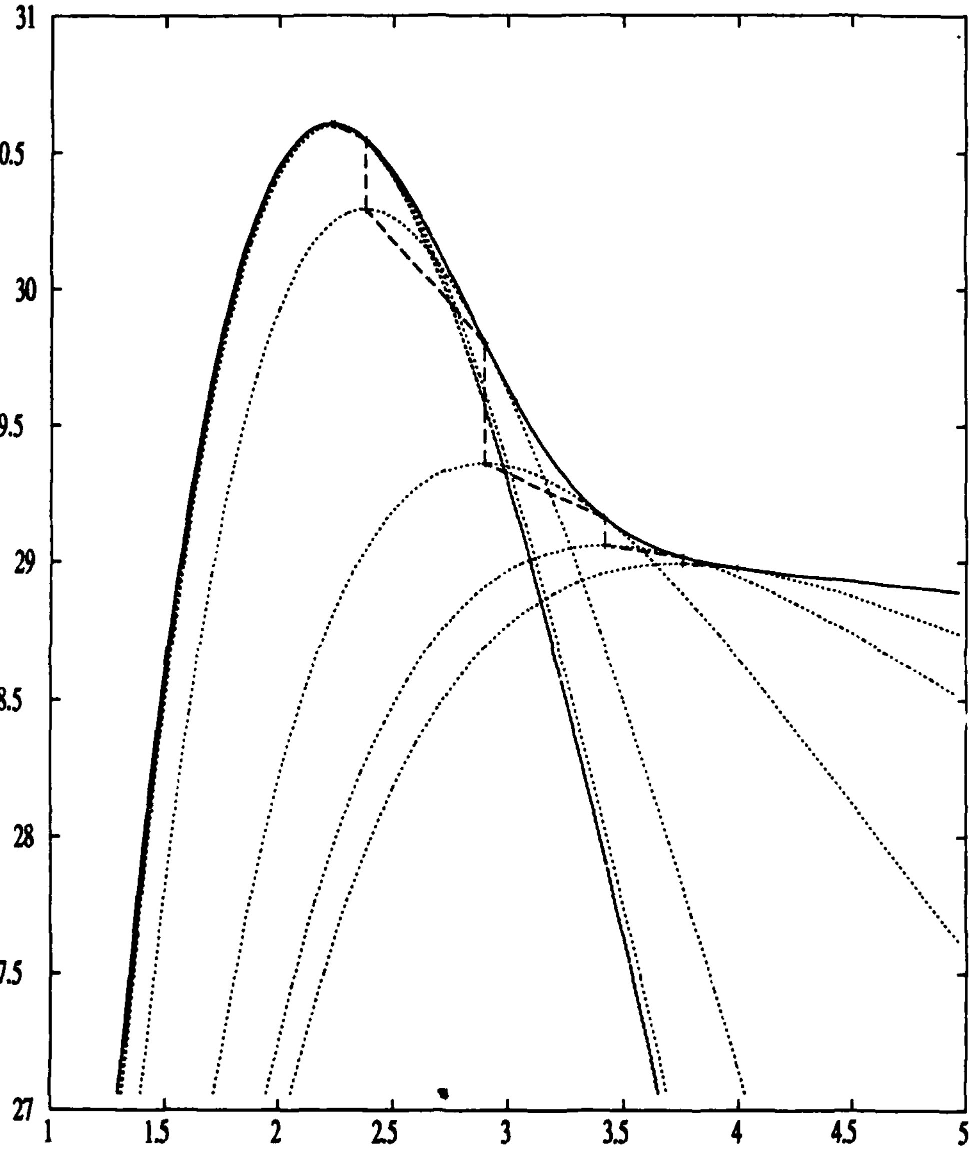

Figure 10: Case V - EM algorithm

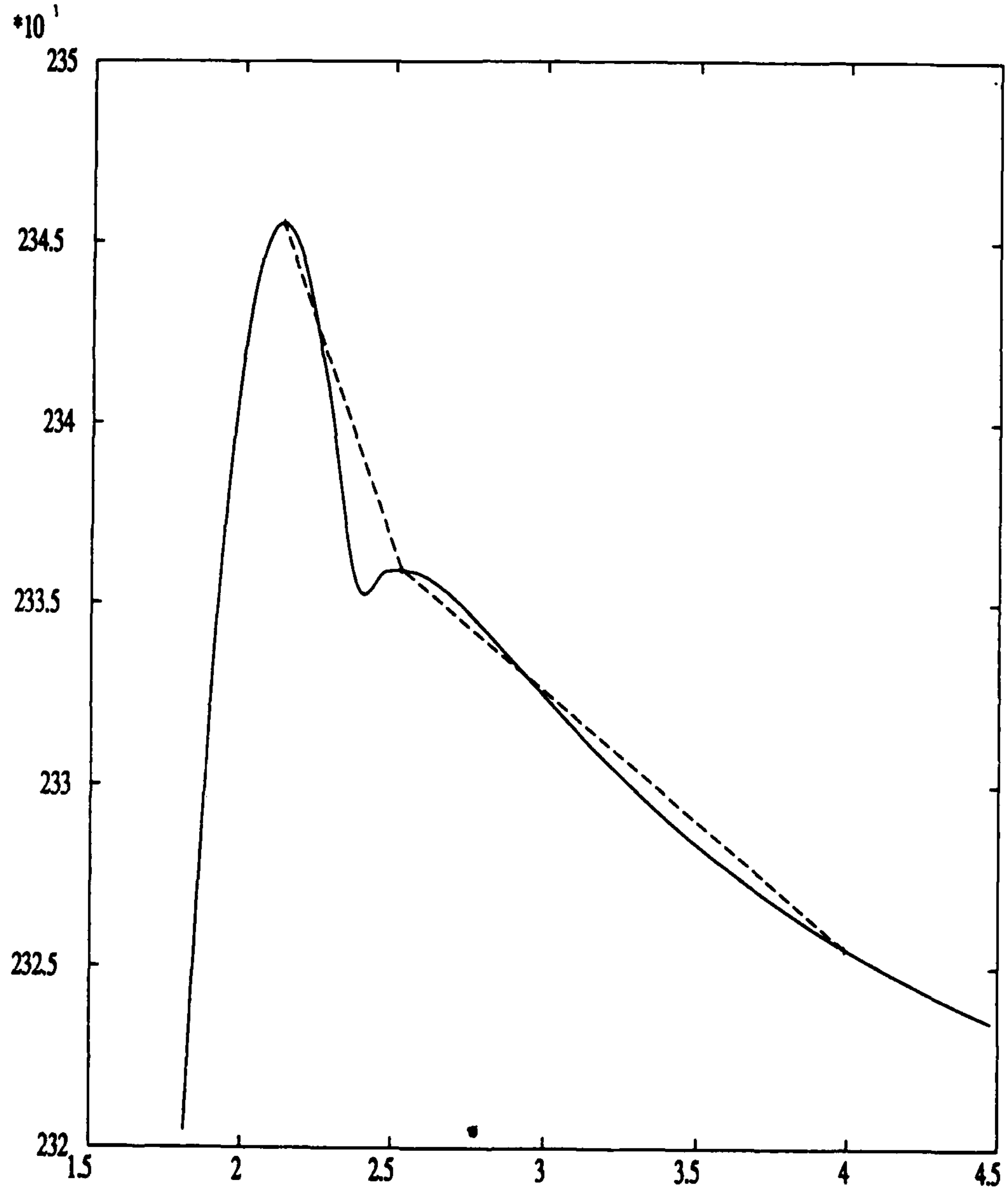

Figure 11: Case VI – Direct maximization

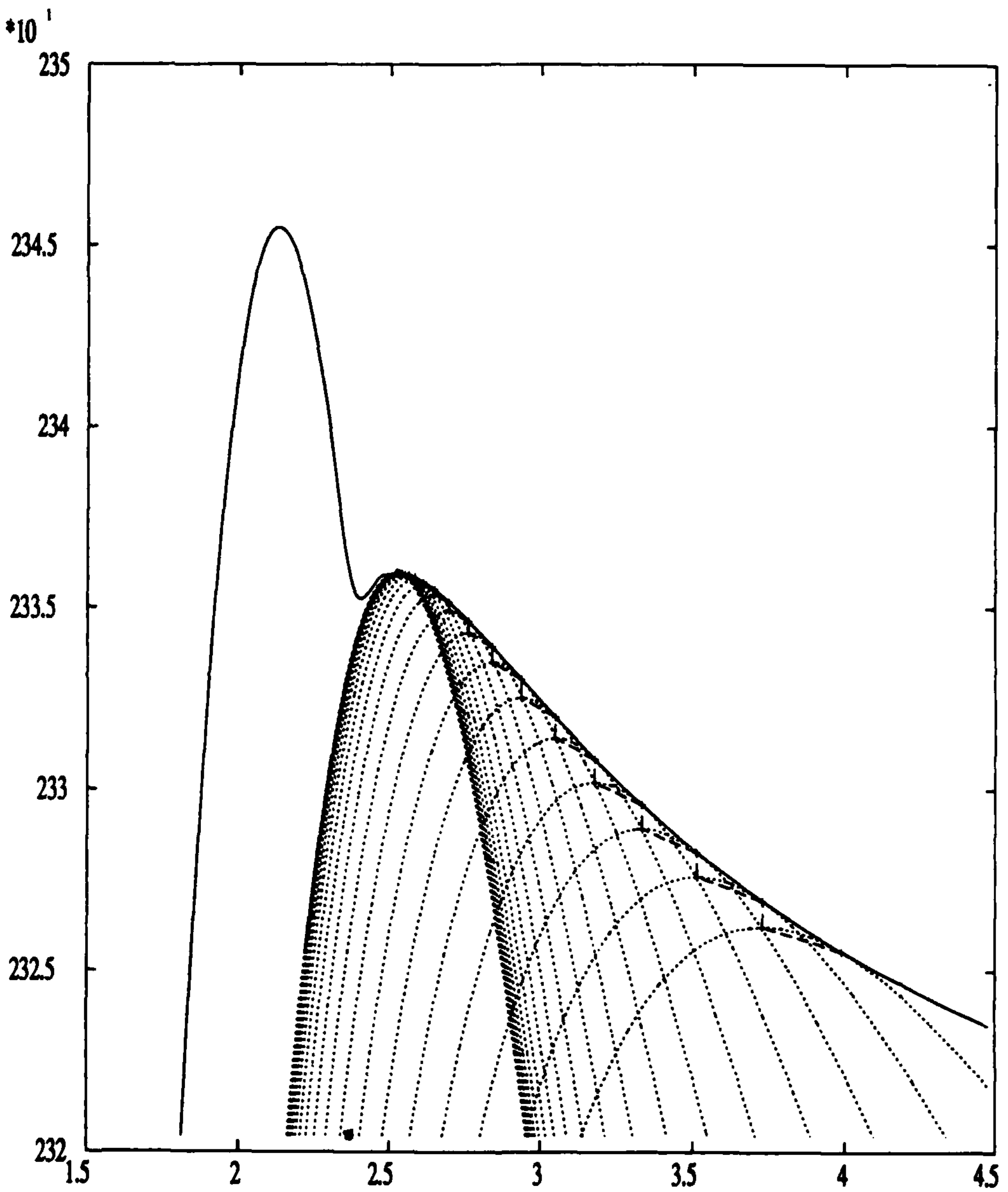

Figure 12: Case VI - EM algorithm

Un théorème d'unicité pour l'équation de Zakaï.

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Résumé : Nous montrons, sous des hypothèses assez générales, que la densité conditionnelle non normalisée en filtrage non linéaire est l'unique solution - dans un espace convenable de processus - de l'équation de Zakaï. La principale restriction est que tous les coefficients doivent être bornés.

Abstract : We prove, under rather general conditions, that the conditional density in nonlinear filtering is the unique solution - within an appropriate space of processes - of Zakaï's equation. The main restriction is that all coefficients are supposed to be bounded.

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Introduction

Il est bien connu que dans un problème de filtrage non linéaire de processus de diffusion, une certaine version de la loi conditionnelle non normalisée satisfait une équation aux dérivées partielles stochastique linéaire appelée équation de Zakaï - cf. ci-dessous. Il est dès lors intéressant de caractériser cette loi conditionnelle non normalisée comme étant l'unique solution - en un certain sens - de cette équation de Zakaï. Un tel résultat peut se décomposer en deux parties : d'une part un théorème d'unicité pour l'équation de Zakaï dans une certaine classe $\mathcal{C}$ de processus, et d'autre part un résultat de régularité permettant d'affirmer que la loi conditionnelle non normalisée appartient à cette même classe $\mathcal{C}$.

De nombreux résultats de ce type ont été établis par divers auteurs, dans des cadres plus ou moins généraux. Dans le cas de coefficients bornés, Kunita [11] et Szpirglas [17] ont établi un résultat d'unicité dans le cas où le signal et le bruit d'observation sont indépendants, Krylov-Rosovskii [10] et Pardoux [13] sous des hypothèses d'uniforme ellipticité. Divers types de coefficients non bornés ont été considérés dans le cas où le signal et le bruit d'observation sont indépendants dans Pardoux [15], Baras-Blankenship-Hopkins [1], Fleming-Mitter [6], Kallianpur-Karandikar [10], Ferreyra [5] et Kurtz-Ocone [13] (Dans ce dernier article, sont également traités des cas où il y a corrélation des bruits). Bensoussan [2] considère des coefficients non bornés dans des cas où le signal et l'observation sont corrélés avec une condition d'ellipticité. Haussmann [9] considère des coefficients dépendant de l'observation. Enfin Canarsa-Vespri [3] considèrent le problème d'unicité pour l'équation de Zakaï avec des coefficients non bornés et une corrélation entre signal et bruit lorsque l'observation est en dimension un. Le fait que cette unique solution est la loi non normalisée est établi par Florchinger [7].

Notre but est d'établir un résultat très général, en supposant cependant tous les coefficients bornés et de classe C^{∞}. Par ailleurs, la loi initiale est quelconque, et nous ne faisons absolument aucune hypothèse de non-dégénérescence : la loi conditionnelle ne possède pas nécessairement de densité. En outre, le signal et le bruit sont corrélés, et tous les coefficients dépendent de tout le passé de l'observation. Ce dernier point est très important pour les applications au contrôle stochastique avec observation partielle.

Nous allons tout d'abord établir un résultat d'unicité d'une équation aux dérivées partielles stochastique dans des espaces de Sobolev d'indice quelconque, à l'aide de propriétés élémentaires des opérateurs pseudo-différentiels. Ensuite nous montrerons que la loi conditionnelle non normalisée appartient à un certain

espace de Sobolev d'indice négatif.

1. Position du problème

Nous allons étudier l'unicité des solutions d'une équation aux dérivées partielles du type suivant (on utilise ici et dans toute la suite la convention de sommation sur indice répété) :

$$(*) \qquad du_t = Au_t \, dt + B_i u_t \, dw_t^i \ , \quad u_0 \text{ donné}$$

où :

(i) $\{w_t\}$ est un processus de Wiener sur un espace de probabilité $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ à valeurs dans $\mathbb{R}^p$.

(i) A (resp $B_1, \ldots, B_p$) sont des opérateurs différentiels sur $\mathbb{R}^n$ d'ordre 2 (resp.1) s'écrivant :

$$A = \frac{1}{2} \sum_{i=1}^{m} X_i^2 + X_0 + c + \frac{1}{2} \sum_{i=1}^{P} B_i^2$$

$$B_i = Y_i + h_i$$

où X_0, $X_1 \ldots X_m, Y_1 \ldots Y_p$ sont des champs de vecteurs sur $\mathbb{R}^n$, $c, h_1 \ldots h_p$ des fonctions sur $\mathbb{R}^n$; dépendant de $(\omega, t) \in \Omega \times \mathbb{R}_+$.
Si α désigne l'une de ces fonctions ou l'un des coefficients des champs de vecteurs, on suppose que :

- $\alpha : \Omega \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ est $\mathcal{P} \otimes B_n$ mesurable, où $\mathcal{P}$ désigne la tribu des événements progressivement mesurables de $\Omega \times \mathbb{R}_+$, et B_n la tribu borélienne de $\mathbb{R}^n$.
- pour tout $(\omega, t) \in \Omega \times \mathbb{R}_+$, $\alpha(\omega, t, .)$ est dans $C_b^\infty(\mathbb{R}^n)$, les bornes étant uniformes en (ω, t).

1.1. Classes de processus à valeurs dans des espaces de Sobolev

Dans ce paragraphe, nous rappelons les principales définitions concernant les espaces de Sobolev et les opérateurs pseudo-différentiels. Nous renvoyons au livre de Trèves [18]*pour un exposé détaillé sur ce sujet.

1.1.1. Espaces de Sobolev

Comme dans [4], on introduit, pour α dans $\mathbb{R}$, le potentiel de Bessel Λ_α qui agit sur $S'(\mathbb{R}^n)$ (espace des distributions tempérées) par la formule suivante :

$$\widehat{\Lambda_\alpha f}(\xi) = (1+|\xi|^2)^{\alpha/2} \, \hat{f}(\xi).$$

Alors $H^{\alpha}(\mathbb{R}^n) = \left\{f\in S'(\mathbb{R}^n), \Lambda_\alpha f \in L^2(\mathbb{R}^n)\right\}$ et, si $f\in H^{\alpha}(\mathbb{R}^n)$, $\|f\|_\alpha = \|\Lambda_\alpha f\|_0$ où $\| \ \|_0$ est la norme usuelle dans $L^2(\mathbb{R}^n)$ ((,) désigne le produit scalaire associé).

Pour $u\in H^{\alpha+1}(\mathbb{R}^n), v\in H^{\alpha-1}(\mathbb{R}^n)$, on pose :

$$<u,v>_\alpha = (\Lambda_{\alpha+1} u \ , \ \Lambda_{\alpha-1} v) \ ;$$

$<.,.>_\alpha$ est donc le produit de dualité entre $H^{\alpha+1}(\mathbb{R}^n)$ et $H^{\alpha-1}(\mathbb{R}^n)$ lorsque ce dernier espace est identifié au dual de $H^{\alpha+1}(\mathbb{R}^n)$, ce qui correspond à identifier $H^{\alpha}(\mathbb{R}^n)$ à son dual.

1.1.2. Opérateurs pseudo-différentiels

Définition : Soit U un ouvert de $\mathbb{R}^n$ et $m\in\mathbb{R}$: on appelle amplitude d'ordre m sur U une application C^∞ $a : U\times\mathbb{R}^n \to \mathbb{C}$ telle que, pour tout compact K de U et tout couple de n-uplets (α,β), il existe une constante $C_{\alpha,\beta}(K)$ telle que :

$$\sup_{\substack{x\in K \\ \xi\in\mathbb{R}^n}} |D_\xi^\alpha D_x^\beta a(x,\xi)| \leq C_{\alpha,\beta}(K)(1+|\xi|)^{m-|\alpha|}.$$

L'ensemble des amplitudes d'ordre m sur U est noté $S_m(U)$.

notations : $\mathcal{D}'(U)$ est l'espace des distributions sur U, i.e. le dual de $C_c^\infty(\Omega)$; $\mathcal{E}'(U)$ est l'espace des distributions à support compact sur U, i.e. le dual de $C^\infty(U)$.

Définition : On appelle opérateur pseudo-différentiel standard d'ordre m sur U une application linéaire $A : \mathcal{E}'(U) \to \mathcal{D}'(U)$ de la forme :

$$Au(x) = (2\Pi)^{-n} \int e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi)d\xi$$

où $a\in S_m(U)$; a est le symbole de A.
L'ensemble des opérateurs pseudo-différentiels standard d'ordre m sur U est noté $\Psi_m(U)$.

Exemples :

. Le potentiel de Bessel Λ_α est un opérateur pseudo différentiel d'ordre α.

. Soit P un polynôme à n variables complexes d'ordre $m\in\mathbb{N}$. $A=P(-i\frac{\partial}{\partial x})$ est un opérateur différentiel d'ordre m qui agit sur une distribution $u\in\mathcal{E}'(U)$ par la formule :

$$Au(x) = (2\Pi)^{-n} \int e^{ix\cdot\xi} P(\xi)\, \hat{u}(\xi)\, d\xi$$

c'est donc un opérateur pseudo-différentiel d'ordre m, de symbole P.

. Si A est un opérateur différentiel à coefficients non constants, on peut montrer qu'il existe un opérateur pseudo-différentiel B tel que A-B soit un opérateur régularisant i.e. d'ordre $-\infty$. On peut donc identifier A à un élément de $\dot{\Psi}_m(U) = \Psi_m(U)/\Psi_{-\infty}(U)$.

<u>Proposition</u> : Un opérateur pseudo-différentiel d'ordre m sur $U \subset \mathbb{R}^n$ définit une application continue de $H^s_c(U)$ dans $H^{s-m}_{loc}(U)$ où

$$H^s(U) = \{u \in H^s(\mathbb{R}^n),\ \operatorname{supp} u \subset U\}$$
$$H^s_c(U) = H^s(U) \cap \mathcal{E}'(U)$$
$$H^s_{loc}(U) = \left\{u \in \mathcal{S}'(\mathbb{R}^n),\ \forall \varphi \in C^\infty_c(U),\ \varphi u \in H^s(U)\right\}$$

<u>Définition</u> : Soit $\dot{A} \in \dot{\Psi}^m(U)$ et $\dot{B} \in \dot{\Psi}^{m'}(\Psi)$. Alors $[\dot{A},\dot{B}] = \dot{A}\dot{B}-\dot{B}\dot{A}$ est dans $\dot{\Psi}^{m+m'-1}$.

$\dot{\Psi}(U) = \bigcup_{m \in \mathbb{N}} \dot{\Psi}^m(U)$ muni de ce crochet est une algèbre de Lie.

Dans la suite de l'article, nous travaillerons sur la sous algèbre de Lie engendrée par les potentiels de Bessel et les opérateurs différentiels.

<u>1.1.3. Classes de processus</u>

On définit la classe de processus (cf. [4] et [14]) ;

$$\mathcal{H}^\alpha_T = L^2(U \times [0,T], H^\alpha(\mathbb{R}^n))$$

et, sur cet espace, la norme

$$|||u|||_{\alpha,T} = \left\{E\left(\int_0^T \|u(\omega,t)\|^2_\alpha\, dt\right)\right\}^{1/2}$$

On introduit aussi : $\mathcal{K}^\alpha_T = \mathcal{H}^{\alpha+1}_T \cap L^2(U\ ;\ C([0,T]\ ;\ H^\alpha(\mathbb{R}^n)))$

<u>1.2. Théorème</u>

Soit $\alpha \in \mathbb{R}$. L'équation (*) admet au plus une solution dans $\bigcap_{T>0} \mathcal{K}^\alpha_T$ dont la valeur en t=0 soit un élément donné de $H^\alpha(\mathbb{R}^n)$.

<u>2. Démonstration du théorème.</u>

Elle se décompose en cinq étapes.

2.1 Formule de Itô.

2.1.1. Proposition : Soit u une solution de (*) dans $\mathcal{W}_T^\alpha$

Alors $v=\Lambda_\alpha u$ est dans $\mathcal{W}_T^0$ et vérifie :

$$\|v_t\|_0^2 = \|v_0\|_0^2 + \int_0^t \left\{ 2<v_s, \Lambda_\alpha A\, u_s>_0 + \sum_{i=1}^p \|B_i u_s\|_\alpha^2 \right\} ds$$

$$+ 2\int_0^t (v_s, \Lambda_\alpha B_i u_s) dw_s^i, \forall o \leq t \leq T.$$

Preuve : Si u est dans $\mathcal{H}_T^{\alpha+1}$, d'après les hypothèses faites sur les opérateurs $A, B_1 \ldots B_p$, Au est dans $\mathcal{H}_T^{\alpha-1}$ et $B_i u$ dans $\mathcal{H}_T^\alpha$. Donc $\Lambda^\alpha u$ est dans $\mathcal{H}_T^1$, Λ^α Au dans $\mathcal{H}_T^{-1}$ et $B_i u$ dans $\mathcal{H}_T^0$. On a, de plus, l'égalité :

$$\Lambda^\alpha u_t = \Lambda^\alpha u_0 + \int_0^t \Lambda^\alpha A u_0 \, ds + \int_0^t \Lambda^\alpha B_i u_s dw_s^i \ , \ \forall o \leq t \leq T.$$

En effet ceci revient à montrer que l'on peut commuter Λ^α avec l'intégration de Lebesgue ou stochastique, ce qui est obtenu en approchant Au (resp. $B_i u$) par des fonctions simples dans $\mathcal{H}_T^{\alpha-1}$ (resp. $\mathcal{H}_T^\alpha$).

La proposition résulte alors de la formule d'Itô de [14] dont les hypothèses sont trivialement vérifiées.

2.1.2. Corollaire :

Soit u une solution de (*) dans $\mathcal{W}_T^\alpha$ alors :

$$E(\|u_t\|_\alpha^2) = E(\|u_0\|_\alpha^2) + \int_0^t E\left\{ 2<u_s, Au_s>_\alpha + \sum_{i=1}^p \|B_i u_s\|_\alpha^2 \right\} ds, \ 0 \leq t \leq T.$$

Dans les paragraphes suivants, on estime les termes sous l'intégrale en fonction de $\|u_s\|_\alpha^2$ afin de pouvoir appliquer le lemme de Gronwall.

2.2. Inégalité à priori dans $H^1(\mathbb{R}^n)$.

2.2.1. Proposition : Il existe une constante K>0 telle que :

$$\forall f \in H^1(\mathbb{R}^n), \ <Af, f>_0 \leq K\|f\|_0^2 + \frac{1}{2}\sum_{i=1}^p <f, B_i^2 f>_0$$

Preuve : Nous commençons par énoncer un lemme que nous utiliserons plusieurs fois par la suite.

2.2.2. lemme : Soit X un champ de vecteurs de classe C_b^1 sur $\mathbb{R}^n$. Alors :

(i) $X^* = -X + b$ où $b \in C_b(\mathbb{R}^n)$

(ii) Il existe une constante K telle que :

$$\forall f \in H^1(\mathbb{R}^n), \ |(Xf,f)| \leq K\|f\|_0^2$$

Preuve : Si $X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}$, $X^* = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(X_i \,.)$

$$= -\sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i} - \sum_{i=1}^{n} \frac{\partial X_i}{\partial x_i} = -X - \operatorname{div} X$$

Soit $f \in H^1(\mathbb{R}^n)$: $(Xf,f) = (f,X^*f) = -(f,Xf) - (f,\operatorname{div}X.f)$.

D'où $(Xf,f) = -\frac{1}{2}(f,\operatorname{div}X.f)$

Alors, $|(Xf,f)| \leq \frac{1}{2}\|\operatorname{div}X\|_\infty \|f\|_0^2$.

Suite de la preuve de la proposition 2.2.1.
Par définition de A, on a :

$$<Af,f>_0 = \frac{1}{2}\sum_{i=1}^{m} <X_i^2 f,f>_0 + <X_0 f,f>_0 + <cf,f>_0$$

$$+ \frac{1}{2}\sum_{i=1}^{p} <B_i^2 f,f>_0$$

Il résulte alors du lemme 2.2.2. que :

$$<Af,f>_0 \leq \frac{1}{2}\sum_{i=1}^{m} <X_i^2 f,f>_0 + K_1\|f_0\|^2 + \frac{1}{2}\sum_{i=1}^{p} <B_i^2 f,f>_0 .$$

Il reste à étudier le terme $\sum_{i=1}^{m} <X_i^2 f, f>_0$. Soit $i \in \{1,\dots,n\}$:

$$<X_i^2 f, f>_0 = (X_i f, X_i^* f) = -\|X_i f\|_0^2 - (X_i f, \operatorname{div} X_i . f)$$

D'où $$<X_i^2 f, f>_0 \leq K_2 \|f\|_0^2$$

ce qui achève la preuve.

2.3. Inégalité a priori dans $H^\alpha(\mathbb{R}^n)$.

2.3.1. Proposition : Il existe une constante $K>0$ telle que :

$$\forall f \in H^{\alpha+1}(\mathbb{R}^n),\ <Af, f>_\alpha \leq K\|f\|_\alpha^2 + \frac{1}{2}\sum_{i=1}^{p} <\Lambda_\alpha f, B_i^2 \Lambda_\alpha f>_0$$

Preuve : Soit $f \in H^{\alpha+1}(\mathbb{R}^n)$.

$$<Af, f>_\alpha = <\Lambda_\alpha A f, \Lambda_\alpha f>_0 = <A\, \Lambda_\alpha f, \Lambda_\alpha f>_0$$
$$+ <[\Lambda_\alpha, A] f, \Lambda_\alpha f>_0 .$$

Le premier terme est estimé grâce à la proposition 2.1.1..
Afin de majorer le second, on utilise une technique de commutation et passage de l'adjoint.
On introduit
$g = \Lambda_\alpha f$ et $T = [\Lambda_\alpha, A]\Lambda_{-\alpha}$.

$$<[\Lambda_\alpha, A] f, \Lambda_\alpha f>_0 = (Tg, g) = \frac{1}{2}((T+T^*)g, g).$$

Calculons T^* :

$$T^* = \Lambda_{-\alpha}[A^*, \Lambda_\alpha] = [\Lambda_\alpha, -A^*]\, \Lambda_{-\alpha} + [\Lambda_{-\alpha}, [A^*, \Lambda_\alpha]].$$

Or $A^* - A$ est un opérateur d'ordre 1. En effet :

$$A^* - A = \frac{1}{2}\sum_{i=1}^{m} (X_i^{*2} - X_i^2)^* + X_0^* - X_0$$

$$= \frac{1}{2}\sum_{i=1}^{m} \left\{(-X_i - \operatorname{div} X_i)^2 - X_i^2\right\} - 2X_0 - \operatorname{div} X_0$$

$$= \sum_{i=1}^{m} \operatorname{div}X_i . X_i - 2X_0 + \frac{1}{2}\sum_{i=1}^{m} (\operatorname{div}X_i)^2 - \operatorname{div}X_0$$

d'où

$$T+T^* = [\Lambda_\alpha, A-A^*]\,\Lambda_{-\alpha} + [\Lambda_{-\alpha}, [A^*, \Lambda_\alpha]]$$

est un opérateur d'ordre 0,; ce qui permet de conclure.

2.4. Inégalité d'énergie dans $H^\alpha(\mathbb{R}^n)$.

2.4.1. Proposition : Il existe une constante K>0 telle que :

$$\forall f \in H^{\alpha+1}(\mathbb{R}^n),\ \sum_{i=1}^{p} \|B_i f\|_\alpha^2 \leqslant - \sum_{i=1}^{p} \langle \Lambda_\alpha f, B_i^2 \Lambda_\alpha f\rangle_0 + K\|f\|_\alpha^2$$

Preuve : Commençons par le cas $\alpha=0$. On a, pour $i \in \{1...p\}$:

$$\langle B_i f, B_i f\rangle_0 = \langle f, B_i^* B_i f\rangle_0 = -\langle f, B_i^2 f\rangle_0 - (f, \operatorname{div}B_i . B_i f)$$

Le dernier terme est estimé grâce au lemme 2.2.2..
Soit α un réel quelconque

$$\|B_i f\|_\alpha^2 = (\Lambda_\alpha B_i f, \Lambda_\alpha B_i f) = (B_i \Lambda_\alpha f, B_i \Lambda_\alpha f)$$

$$+ 2 \langle B_i \Lambda_\alpha f, [\Lambda_\alpha, B_i] f\rangle_0 + \|[\Lambda_\alpha, B_i] f\|_0^2 .$$

L'opérateur $[\Lambda_\alpha, B_i]$ étant d'ordre α, le dernier terme est estimé aisément. D'autre part, d'après l'étude du cas $\alpha=0$, le premier terme est majoré par :

$$-\langle \Lambda_\alpha f, B_i^2 \Lambda_\alpha f\rangle_0 + K\|f\|_\alpha^2$$

Il reste donc à estimer : $\langle \Lambda_\alpha f,\ B_i^* [\Lambda_\alpha, B_i] f\rangle_0 = \langle \Lambda_\alpha f, \tilde{T} \Lambda_\alpha f\rangle_0$

avec $\tilde{T} = B_i^* [\Lambda_\alpha, B_i] \Lambda_{-\alpha}$

On raisonne comme dans la preuve de la proposition 2.3.1. ie on écrit : $\langle \Lambda_\alpha f, \tilde{T} \Lambda_\alpha f\rangle_0 = \frac{1}{2} \langle \Lambda_\alpha f, (\tilde{T}+\tilde{T}^*) \Lambda_\alpha f\rangle_0$

et on va montrer que $\tilde{T}+\tilde{T}^*$ est un opérateur d'ordre 0.

$$\tilde{T}^* = \Lambda_{-\alpha}[B_i^*, \Lambda_\alpha] B_i = [\Lambda_{-\alpha}, [B_i^*, \Lambda_\alpha]] B_i$$

$$+ [[B_i^*, \Lambda_\alpha]\Lambda_{-\alpha}, B_i] + B_i\ [B_i^*, \Lambda_\alpha]\Lambda_{-\alpha}$$

D'où : $$\tilde{T}^* = B_i^* [\Lambda_\alpha, B_i^*]\Lambda_{-\alpha} + \mathrm{div}B_i [\Lambda_\alpha, B_i^*]\Lambda_{-\alpha}$$

$$+ \text{opérateur d'ordre } 0$$

et $\tilde{T}+\tilde{T}^* = B_i^* [\Lambda_\alpha, -\mathrm{div}B_i]\Lambda_{-\alpha}$ + opérateur d'ordre 0 est un opérateur d'ordre 0.

2.5. Fin de la démonstration

Il résulte du corollaire 2.1.2., des propositions 2.3.1. et 2.4. l'inégalité suivante : si u est une solution de (*) dans $\mathcal{H}^{\alpha+1}$, alors :

$$E(\|u_t\|_\alpha^2) \leq E(\|u_0\|_\alpha^2) + K\int_0^t E(\|u_s\|_\alpha^2)ds.$$

On déduit alors du lemme de Gronwall que :

$$E(\|u_t\|_\alpha^2) \leq e^{Kt}\, E(\|u_0\|_\alpha^2)$$

d'où $$|||u|||_{\alpha,T}^2 \leq K'\, E(\|u_0\|_\alpha^2) \quad , \quad \forall T\in\mathbb{R}_+ .$$

On en déduit l'unicité de la solution de (*) dans $\mathcal{H}_T^{\alpha+1}$ pour tout $T\in\mathbb{R}_+$ et par suite dans $\bigcap_{T>0} \mathcal{H}_T^\alpha$.

3. Application au filtrage

3.1. Description du modèle

Considérons le couple (signal, observation) noté (x_t, y_t) solution du système suivant :

$$(F)\begin{cases} dx_t = X_0(t,y,x_t)dt + X_i(t,y,x_t)\circ dw_t^i + \tilde{X}_j(t,y,x_t)(d\tilde{w}_t^j + h^j(t,x_t)dt). \\ dy_t = h(t,y,x_t)dt + d\tilde{w}_t . \\ \quad (x_0, y_0) \text{ suit une loi } \Pi_0\otimes\delta_0 \text{ et est indépendant de } (v,w). \end{cases}$$

On désigne par $\alpha(t,y,x)$ l'une quelconque des composantes des champs de vecteurs ou des fonctions intervenant dans ce système et on fait les hypothèses suivantes :

(i) $$\alpha : \mathbb{R}_+ \times C(\mathbb{R}_+,\mathbb{R}^p)\times\mathbb{R}^n \longrightarrow \mathbb{R}$$

est $\mathcal{P}_y \otimes B_a$ mesurable où $\mathcal{P}_y$ désigne la tribu des événements progressivement mesurables de $\mathbb{R}_+ \times C(\mathbb{R}_+ ; \mathbb{R}^p)$.

(ii) Pour tout couple $(t,y) \in \mathbb{R}_+ \times C(\mathbb{R}_+, \mathbb{R}^p)$, l'application $x \in \mathbb{R}^n \longrightarrow \alpha(t,y,x)$ est de classe $C_b^\infty(\mathbb{R}^n)$, les bornes de α et de ses dérivées étant indépendantes de (t,y).

Notons que sous les conditions (i) et (ii), le système différentiel stochastique (F) possède une unique solution faible (autrement dit, le problème de martingales associé est bien posé). En effet, si $(w_t, y_t \; ; \; t \geq 0)$ est le processus canonique de $\Omega = C(\mathbb{R}_+ ; \mathbb{R}^m) \times C(\mathbb{R}_+ ; \mathbb{R}^p)$ muni de la tribu borélienne et de la mesure de Wiener, l'équation différentielle stochastique :

$$dx_t = X_0(t,y,x_t)dt + X_i(t,y,x_t) \circ dw_t^i + \tilde{X}_j(t,y,x_t)dy_t^j$$

avec x_0 donné dans $\mathbb{R}^n$ possède une unique solution forte. Une application standard du théorème de Girsanov permet de conclure.

Soit Y_t la tribu engendrée par $y_s, s \leq t$.

3.2. L'équation de Zakaï

Rappelons que nous associons au système de filtrage précédent une mesure de probabilité P_0 dite de référence définie par

$$\frac{dP}{dP_0} \Big|_{\mathcal{F}_t} = L_t = \exp\left(\int_0^t h_i(s,y,x_s)dy_s^i - \frac{1}{2}\int_0^t |h(s,y,x_s)|^2 ds\right).$$

On peut alors exprimer le filtre $\Pi_t f = E(f(x_t) | Y_t)$ associé à une fonction f mesurable bornée à l'aide du filtre non normalisé $\rho_t f = E_0(f(x_t)L_t | Y_t)$ par la formule de Kallianpur-Striebel :

$$\Pi_t f = \frac{\rho_t f}{\rho_t 1}.$$

ρ_t est solution au sens des distributions de l'équation de Zakaï :

$$\text{(Z)} \qquad d\rho_t = L_0^* \rho_t dt + L_i^* \rho_t dy_t^i \quad , \qquad \rho_0 = \Pi_0$$

avec :

$$L_0 = \frac{1}{2}\left(\sum_{i=1}^m X_i^2 + \sum_{i=1}^p \tilde{X}_i^2\right) + X_0 + h_i \tilde{X}^i - \frac{1}{2}\sum_{i=1}^p \partial_j \tilde{X}_i \tilde{X}_i^j$$

$$L_i = \tilde{X}_i + h_i$$

où $$\partial_j \tilde{X}_i \tilde{X}_i^j = \sum_{j=1}^{n} \frac{\partial \tilde{X}_i}{\partial x_j} \tilde{X}_i^j$$

3.2.1. Théorème

Soit $\alpha\in\mathbb{R}$. (Z) admet au plus une solution dans $\bigcap_{T>0} \mathcal{W}_T^\alpha$ dont la valeur en t=0 soit donnée dans H^α.

Preuve : Les hypothèses du Th. 1.2. sont clairement satisfaites.

3.2.2. Théorème : Quelle que soit la mesure de probabilité Π_0 sur $\mathbb{R}^n$, et quel que soit $\varepsilon>0$, le filtre non normalisé associé est l'unique solution de (Z) dans $\bigcap_{T>0} \mathcal{W}_T^{-\frac{n}{2}-\varepsilon}$.

Preuve : On va montrer que ρ_0 est dans $H^{-\frac{n}{2}-\varepsilon}$ et ρ dans $\mathcal{K}_T^{-\frac{n}{2}-\varepsilon}$. L'appartenance de ρ à $L^2(\Omega;C([0,T];H^{-\frac{n}{2}-\varepsilon-1}))$ découle alors, d'après le lemme 1.4. de [14], du fait que ρ satisfait (Z) (cf. Pardoux [16] où l'équation (Z) est établie sous des hypothèses différentes des nôtres, mais la même démarche est applicable ici). Nous allons commencer par montrer que, pour tout $t\in[0,T]$, ρ_t est dans $H^{-\frac{n}{2}-\varepsilon}$ p.s..

$$\|\rho_t\|^2_{-\frac{n}{2}-\varepsilon} = \int_{\mathbb{R}^n} (1+|\xi|^2)^{-\frac{n}{2}-\varepsilon} |\widehat{\rho_t}(\xi)|^2 d\xi$$

$$\leqslant \sup_{\xi\in\mathbb{R}^n} |\widehat{\rho_t}(\xi)|^2 \int_{\mathbb{R}^n} (1+|\xi|^2)^{-\frac{n}{2}-\varepsilon} d\xi$$

or $\widehat{\rho_t}(\xi) = E_0(e^{i\xi x_t} L_t|Y_t)$. D'où $|\widehat{\rho_t}(\xi)| \leqslant E_0(L_t|Y_t)$.

En intégrant par rapport à dP dt, on obtient :

$$|||\rho|||^2_{-\frac{n}{2}-\varepsilon,T} \leq C\ T\ E_0(\sup_{t\leq T} L_t^2) < +\infty$$

<u>Note</u> : Nous venons de prendre connaissance d'un article de Fujita [8] où un résultat d'unicité est obtenu par des techniques semblables aux nôtres, mais sous des hypothèses plus restrictives.

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Piecewise linear filtering with small observation noise

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Abstract: We consider a piecewise linear filtering problem with small observation noise. It is shown that one can construct an approximate finite dimensional filter which uses a bunch of Kalman filters, together with a test procedure to decide which Kalman filter to follow.

1. Introduction

The aim of this paper is to propose an approximate optimal filter for the filtering problem:

$$\begin{aligned} dx_t &= f(x_t)\, dt + dw_t \\ dy_t &= h(x_t)\, dt + \varepsilon dv_t \end{aligned}$$

where $\{x_t\}$ is a scalar unobserved process, $\{y_t\}$ is a scalar observed process, ε is a "small" parameter, $\{w_t\}$ and $\{v_t\}$ are mutually independent standard Wiener processes. We assume that $I\!R = \cup_{i=1}^{l} I_i$, where $I_1, \ldots, I_l$ are disjoint intervals, f and h are continuous mappings from $I\!R$ into $I\!R$, whose restrictions to each K_i are affine.

Roughly speaking, our result is a follows. Provided a certain "detectability hypothesis" is satisfied, an approximate optimal filter is given by one of a set of l Kalman filters, the decision about which Kalman filter to follow for a given period of time being taken in view of the outputs of the l Kalman filters.

Let us sketch the general ideas on a simple example. Suppose that $l = 2, I_1 = I\!R_-$ and $I_2 = I\!R_+$, and that:

$$f(x) = \begin{cases} F_+ x, & \text{if } x \geq 0 \\ F_- x, & \text{if } x \leq 0 \end{cases}$$

$$h(x) = \begin{cases} H_+ x, & \text{if } x \geq 0 \\ H_- x, & \text{if } x \leq 0. \end{cases}$$

We now consider the two linear filtering problems:

$$(1_+) \qquad \begin{cases} dx_t = F_+ x_t\, dt + dw_t \\ dy_t = H_+ x_t\, dt + \varepsilon dv_t \end{cases}$$

$$(1_-) \qquad \begin{cases} dx_t = F_- x_t\, dt + dw_t \\ dy_t = H_- x_t\, dt + \varepsilon dv_t \end{cases}$$

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to which one associates two Kalman filters (KF_+) and (KF_-), with outputs (x_t^+, R_t^+) and (x_t^-, R_t^-).

If $H_+H_- > 0$, then h is one to one, and since ε is small, we can almost deduce from $\{y_s, s \leq t\}$ the current value of $h(x_t)$, hence of x_t. More precisely, from the results of Picard [8],[9],[10] (see also Katzur-Bobrovsky-Schuss [6], Bensoussan [1], Ji [5]), we know that the conditional law of x_t, given $\mathcal{Y}_t = \sigma\{y_s; 0 \leq s \leq t\}$ has a small variance.

If for instance $x_s^+ > 0$ and is significantly different from zero for any $s \in [t-\alpha, t]$ (in which case the same is true for x_s^-), the conditional law of x_t given $\mathcal{Y}_t$ is almost completely concentrated on $\mathbb{R}_+$ (at least with probability almost one) and consequently the output of (KF_+) is very close to the conditional law of x_t, given $\mathcal{Y}_t$ (as we will see below, the way in which (KF_+) is initialized does not play a sinificant role), at least with probability almost one.

Suppose now that $H_+H_- < 0$. Then we need some "detectability hypothesis". Indeed, if $f(x) \equiv 0$ and $h(x) = |x|$, then clearly the conditional law of x_t given $\mathcal{F}_t$ is symmetric with respect to 0, and cannot be reasonably approximated by the output of a Kalman filter. Suppose moreover that $|H_+| \neq |H_-|$. Then, for $\varepsilon = 0$, the quadratic variation of $dy_t/dt = h(x_t)$ tells us whether $x_t < 0$ or $x_t > 0$. One may then expect that for $\varepsilon > 0$ but small, the conditional law of x_t given $\mathcal{Y}_t$ has again a small variance, and that a decision about which of (KF_+) or (KF_-) to follow might be reached by comparing the outputs of these two filters. The proof of these facts is the crucial step in our argumentation.

Our results are illustrated by the numerical results in Fleming et all [3]. Let us insist upon the fact that the hypothesis of a high signal-to-noise ratio is crucial for the validity of the algorithm which we propose. Without that hypothesis, the conditional law would spread out over the whole real line, and probably none of the Kalman filters would give an acceptable approximation of the conditional law. A totally different algorithm is proposed for that situation in Pardoux-Savona [7].

Generalisations to higher dimensional situations, as well as to the case where f and h are nonlinear and h piecewise one to one, wil be considered elsewhere.

The paper is organised as follows. In section 2, we formulate precisely the problem and the assumptions, as well as some technical results which will be needed in the sequel. Section 3,4 and 5 study in detail the case where $l = 2, I_1 = \mathbb{R}_-, I_2 = \mathbb{R}_+$, and h is not globally one-to-one and satisfies a "detectability hypothesis". In section 6, we summarize an approximate filtering procedure for the case studied in the previous sections, and indicate the procedure in the general case.

2. Formulation of the problem and preliminary lemmas.

Let $\Omega = C(\mathbb{R}_+) \times C(\mathbb{R}_+)$, $\mathcal{F}$ its Borel field, and $x_t(\omega) = \omega_1(t), y_t(\omega) = \omega_2(t)$. Let P^ε be a probability measure on $(\Omega, \mathcal{F})$ which is such that:

$$x_t = x_0 + \int_0^t f(x_s)\,ds + w_t \tag{2.1}$$

$$y_t = \frac{1}{\varepsilon}\int_0^t h(x_s)\,ds + v_t^\varepsilon, \tag{2.2}$$

where $\{w_t\}$ and $\{v_t^\varepsilon\}$ are two mutually independent standard Wiener processes, x_0 is a random variable independent of $\{w_t, v_t^\varepsilon; t \geq 0\}$ with $E^\varepsilon[exp(cx_0^2)] < \infty$ for some $c > 0$; f and h are continuous mappings from $\mathbb{R}$ into $\mathbb{R}$, which have the following special from. We assume that $\mathbb{R} = \cup_{i=1}^l I_i$, where $I_1, \ldots, I_l$ are closed intervals with disjoint interiors, and the restrictions of f and h to each I_i are affine functions, i.e.

$$f(x) = F_i\, x + f_i; \quad x \in I_i, 1 \leq i \leq l$$

$$h(x) = H_i\, x + h_i; \quad x \in I_i, 1 \leq i \leq l$$

where

$$F_1, \ldots, F_l, f_1, \ldots, f_l, H_1, \ldots, H_l, h_1, \ldots, h_l \in \mathbb{R}.$$

It is well known that P^ε exists and is unique, see e.g. Stroock-Varadhan [11]. $\{x_t\}$ is an unobserved process, while $\{y_t\}$ is observed. We define $\mathcal{Y}_t = \sigma\{y_s; 0 \leq s \leq t\}$ and seek to compute at each time t the conditional law of x_t given $\mathcal{Y}_t$. Our aim is in fact to obtain an asymptotic result, as $\varepsilon \to 0$, concerning a finite dimentional filter to be described later.

We will assume throughout the paper that :

$$H_i \neq 0; \quad 1 \leq i \leq l \tag{H1}$$

Let us now formulate a "detectability hypothesis" which will be assumed to hold throughout the paper:

$$\begin{array}{c} \text{For any point } (i,j) \in \{1, \ldots, l\}^2 \text{ s.t.} \\ i \neq j \text{ and } h(I_i) \cap h(I_j) \text{ has a non void interior,} \\ H_i^2 \neq H_j^2 \end{array} \tag{H2}$$

For $i = 1, \ldots, l$, we can consider a Kalman filter (KF_i), which is the optimal filter for the case where :

$$f(x) = F_i x + f_i, h(x) = H_i x + h_i; \; x \in \mathbb{R}.$$

The Riccati equation for the conditional covariance in (KF_i) reads :

$$\frac{dR_t^i}{dt} = 2F_i R_t^i + 1 - \frac{(H_i R_t^i)^2}{\varepsilon^2}$$

This equation has for small ε a unique stable positive invariant solution, equal to :

$$\frac{\varepsilon}{|H_i|}\left(\sqrt{1+\frac{\varepsilon^2 F_i^2}{H_i^2}}+\varepsilon\frac{F_i}{|H_i|}\right).$$

Let us define $K_i = \left(\sqrt{1+\frac{\varepsilon^2 F_i^2}{H_i^2}}+\varepsilon\frac{F_i}{|H_i|}\right)\text{sign}(H_i)$. The optimal Kalman filter associated to the initial law $N(E(x_0),\varepsilon\frac{K_i}{H_i})$ is given by :

$$(KF_i) \qquad \begin{aligned} dx_t^i &= (F_i x_t^i + f_i)\,dt + K_i(dy_t - \frac{1}{\varepsilon}H_i x_t^i\,dt) \\ x_0^i &= E(x_0) \end{aligned}$$

In most of the paper, we will concentrate on the case $l = 2$, in which we will assume, without loss of generality, that :

$$f_1 = f_2 = h_1 = h_2 = 0,$$
$$I_1 = \mathbb{R}_-, I_2 = \mathbb{R}_+.$$

We will then use the notations:

$$I_- = I_1, F_- = F_1, H_- = H_1$$
$$I_+ = I_2, F_+ = F_2, H_+ = H_2$$

Let us close this section with three lemmas. The proof of the first one is easy and is left to the reader.

Lemma 2.1. *Let $U_1,\dots,U_M$ be i.i.d. random variables, with joint law $N(0,\delta)$. Then for any $a > 0$,*

$$P\left(\max_{1\le k\le M}|U_k|\ge a\right)\le 1-(1-e^{-a^2/2\delta})^M$$

Consequently, when $\delta\to 0$ and $M\to\infty$ in such a way that $M\delta = C$,

$$P\left(\max_{1\le k\le M}|U_k|\ge a\right)\le Me^{-a^2/2\delta}$$

Lemma 2.2. *Let $\{\xi_n, n\in\mathbb{N}\}$ be a sequence of i.i.d. random variables. Let $\Phi(u) = E[exp(u\xi_1)]$. Suppose that Φ is finite on a neighbourhood of the origin, and that $\{u;\Phi(u)\le k\}$ is closed for any $k\in\mathbb{R}_+$. Call μ the common mean of the ξ_n's. For any $\theta > 0$, there exists $C > 0$ such that for any $n\in\mathbb{N}$:*

$$P\left(|\frac{1}{n}\sum_{k=1}^{n}\xi_k - \mu| > \theta\right)\le Ce^{-nC}.$$

Proof: This is large deviation estimate, which can be found e.g. in Ellis [2]□

Lemma 2.3. *Let $\{x_t, t\ge 0\}$ denote the solution of (2.1). Then for any $t > 0$, there exists $c > 0$ such that $E\exp[c\sup_{s\le t}x_s^2] < \infty$*

Proof: This is Theorem 5.7.2 in Kallianpur [5]□

3. The case of two intervals with $H_+H_- < 0$. First step.

We shall treat the case $H_+ > 0, H_- < 0$ (i.e. $h(x) \geq 0, \forall x \in \mathbb{R}$) and use the notations $|H| = \sup(H_+, -H_-), |F| = \sup(|F_+|, |F_-|)$. Recall that assumption (H2) is in force. Since we want to decide on which side of 0 x_t is, we first need to find intervals of time on which no zero crossing takes place, at least with conditional probability almost one.

Let $o \leq a \leq b$, $M = [\frac{b-a}{\varepsilon}]$, and for $l = 0, 1, \ldots, M-1$, define:

$$Y_l^\varepsilon = y_{a+(l+1)\varepsilon} - y_{a+l\varepsilon}$$

$$S_l^\varepsilon = \frac{1}{\varepsilon}\int_{a+l\varepsilon}^{a+(l+1)\varepsilon} h(x_s)\, ds$$

$$V_l^\varepsilon = v_{a+(l+1)\varepsilon}^\varepsilon - v_{a+l\varepsilon}^\varepsilon$$

Note that

$$Y_l^\varepsilon = S_l^\varepsilon + V_l^\varepsilon$$

Define moreover the events:

$$B_+(a,b) = \{x_t > 0; a \leq t \leq b\}$$

$$B_-(a,b) = \{x_t < 0; a \leq t \leq b\}.$$

In case when there is no ambiguity, we shall simply write B_+ and B_-.

Choose $c > 0$, and define :

$$C_\varepsilon = \{|Y_l^\varepsilon| \geq c;\ 0 \leq l \leq M-1\}$$

Proposition 3.1. *For any $\varepsilon_0 > 0$, there exists k s.t. for any $\varepsilon \in (0, \varepsilon_0)$,*

$$P^\varepsilon((B_+ \cup B_-)^c / C_\varepsilon) \leq k e^{-k/\varepsilon}.$$

Proof: If $|Y_l^\varepsilon| \geq c$ and $|V_l^\varepsilon| \leq c/2$, then $|S_l^\varepsilon| \geq c/2$, which implies that there exists $t_l \in [a + l\varepsilon, a + (l+1)\varepsilon]$ s.t. $|h(x_{t_l})| \geq c/2$ and $|x_{t_l}| \geq c_1 = c/2|H|$. It follows that on $(B_+ \cup B_-)^c \cap C_\varepsilon$, we must have either :

$$\sup_{0 \leq k \leq M-1} |V_l^\varepsilon| \geq \frac{c}{2}$$

or else

$$\sup_{0 \leq k \leq M-1}\ \sup_{a+l\varepsilon \leq s \leq a+(l+1)\varepsilon} |x_t - x_s| \geq c_1$$

From lemma 2.1, $\forall \varepsilon_0 > 0, \exists c_0 > 0$ s.t., $\forall \varepsilon \in (0, \varepsilon_0)$,

$$P^\varepsilon\left(\sup_{0 \leq l \leq M-1} |V_l^\varepsilon| \geq \frac{c}{2}\right) \leq c_0 e^{-c_0/\varepsilon}$$

Now, for $a + l\varepsilon \leq s \leq t \leq a + (l+1)\varepsilon$,

$$|x_t - x_s| \leq \varepsilon |F| (\sup_{a \leq t \leq b} |x_t|) + |w_t - w_s|$$

Consequently,

$$\{ \sup_{l \leq M-1} \sup_{a+l\varepsilon \leq s \leq t \leq a+(l+1)\varepsilon} |x_t - x_s| \geq c_1 \}$$
$$\subset \{ \sup_{a \leq t \leq b} |x_t| \geq \frac{c_1}{2\varepsilon |F|} \} \cup \{ \sup_{l \leq M-1} \sup_{a+l\varepsilon \leq t \leq a+(l+1)\varepsilon} |w_t - w_{a+l\varepsilon}| \geq \frac{c_1}{2} \}$$

It follows readily from lemma 2.3 that there exists c_2 s.t. :

$$P\left(\sup_{a \leq t \leq b} |x_t| \geq \frac{c_1}{2\varepsilon |F|} \right) \leq c_2 e^{-c_2/\varepsilon^2}$$

Noting that the sequence $\{\sup_{a+l\varepsilon \leq t \leq a+(l+1)\varepsilon} |w_t - w_{a+l\varepsilon}|; a \leq l \leq M-1\}$ is i.i.d., and that :

$$P\left(\sup_{a+l\varepsilon \leq t \leq a+(l+1)\varepsilon} |w_t - w_{a+l\varepsilon}| \geq \frac{c_1}{4} \right) = 2P(|w_\varepsilon| \geq \frac{c_1}{4}),$$

it follows from an argument similar to the proof of Lemma 2.1 that $\forall \varepsilon > 0, \exists c_3$ s.t. $\forall \varepsilon \leq \varepsilon_0$,

$$P\left(\sup_{l \leq M-1} \sup_{a+l\varepsilon \leq t \leq a+(l+1)\varepsilon} |w_t - w_{a+l\varepsilon}| \geq \frac{c_1}{4} \right) \leq c_3 e^{-c_3/\varepsilon}$$

Finally, note that $C_\varepsilon \supset \{h(x_t) \geq 2c, a \leq t \leq b\} \cap \{\sup_{0 \leq l \leq M-1} |V_l^\varepsilon| \leq c\}$, and from independence,

$$P^\varepsilon(C_\varepsilon) \geq P(h(x_t) \geq 2c, a \leq t \leq b) P^\varepsilon(\sup_{0 \leq l \leq M-1} |V_l^\varepsilon| \leq c)$$

so that $\liminf_{\varepsilon \to 0} P^\varepsilon(C_\varepsilon) > 0$□

4. The case of two intervals with $H_+H_- < 0$. Second step.

We now want to show that, once we know that no zero crossing has occured on $[a, b]$ with probability almost one, then we know wheter $\{x_b > 0\}$ or $\{x_b < 0\}$, with a very small probability of error.

For that sake, let us define, with the notations introduced in the last section :

$$Z_\epsilon^o = \frac{1}{b-a} \sum_{0 \le l \le M-1; l \text{ odd}} (Y_{l+1}^\epsilon - Y_l^\epsilon)^2$$

$$Z_\epsilon^e = \frac{1}{b-a} \sum_{0 \le l \le M-1; l \text{ even}} (Y_{l+1}^\epsilon - Y_l^\epsilon)^2$$

On B_+,

$$\begin{aligned}
Y_{l+1}^\epsilon - Y_l^\epsilon =& S_{l+1}^\epsilon - S_l^\epsilon + V_{l+1}^\epsilon - V_l^\epsilon \\
=& \frac{H_+}{\varepsilon} \int_{a+l\epsilon}^{a+(l+1)\epsilon} (w_{s+\epsilon} - w_s)\, ds + V_{l+1}^\epsilon - V_l^\epsilon + \frac{H_+F_+}{\varepsilon} \int_{a+l\epsilon}^{a+(l+1)\epsilon} \int_s^{s+\epsilon} x_u \, duds \\
Z_\epsilon^o =& \frac{1}{b-a} \sum_{l \text{ odd}} \alpha_l^2 + \frac{1}{b-a} \sum_{l \text{ odd}} (\beta_l^2 + 2\alpha_l\beta_l) \\
Z_\epsilon^e =& \frac{1}{b-a} \sum_{l \text{ even}} \alpha_l^2 + \frac{1}{b-a} \sum_{l \text{ even}} (\beta_l^2 + 2\alpha_l\beta_l)
\end{aligned}$$

where we have dropped the dependence in ε, and defined:

$$\begin{aligned}
\alpha_l =& \frac{H_+}{\varepsilon} \int_{a+l\epsilon}^{a+(l+1)\epsilon} (w_{s+\epsilon} - w_s)\, ds + V_{l+1}^\epsilon - V_l^\epsilon \\
\beta_l =& \frac{H_+F_+}{\varepsilon} \int_{a+l\epsilon}^{a+(l+1)\epsilon} \int_s^{s+\epsilon} x_u \, duds
\end{aligned}$$

Note that $\alpha_l \simeq N(0, 2\varepsilon(1 + \frac{H_+^2}{3}))$, and both sequences $\{\alpha_l; l \text{ odd}\}$ and $\{\alpha_l; l \text{ even}\}$ are i.i.d.. Call M_o the number of odd integers in the intervals $[0, M-1]$, and M_e the number of even integers on the same interval, and define $\rho_o = \frac{\epsilon M_o}{b-a}$, $\rho_e = \frac{\epsilon M_e}{b-a}$. Note that ρ_o and ρ_e are both close to $\frac{1}{2}$.

We are going to show that :

Lemma 4.1. *For any θ and $\varepsilon_0 > 0$, there exists $c > 0$ s.t. for any $\varepsilon \in (0, \varepsilon_0)$,*

$$P^\epsilon \left(\{|Z_\epsilon^o - 2\rho_o(1 + \frac{H_+^2}{3})| \ge \theta\} \cap B_+ \right) \le ce^{-c/\sqrt{\epsilon}}$$

$$P^\epsilon \left(\{|Z_\epsilon^e - 2\rho_o(1 + \frac{H_-^2}{3})| \ge \theta\} \cap B_- \right) \le ce^{-c/\sqrt{\epsilon}}$$

and similar estimates hold with Z_ε^o replaced by Z_ε^e, ρ_o by ρ_e.

Let us first see the conclusion which can be drawn from Lemma 4.1. Suppose to fix the ideas that $H_+^2 > H_-^2$. Define:

$$C_+^\varepsilon = C_\varepsilon \cap \left\{ Z_\varepsilon^o \geq \rho_o \left(2 + \frac{H_+^2 + H_-^2}{3} \right) \right\}$$

$$C_-^\varepsilon = C_\varepsilon \cap \left\{ Z_\varepsilon^o < \rho_o \left(2 + \frac{H_+^2 + H_-^2}{3} \right) \right\}$$

Note that $C_+^\varepsilon \cup C_-^\varepsilon = C_\varepsilon$, and $C_+^\varepsilon \cap C_-^\varepsilon = \emptyset$.

Proposition 4.2. *For any $\varepsilon_0 > 0$, there exists k s.t.*

$$P^\varepsilon(B_+^c / C_+^\varepsilon) \leq k e^{-k/\sqrt{\varepsilon}}$$

and

$$P^\varepsilon(B_-^c / C_-^\varepsilon) \leq k e^{-k/\sqrt{\varepsilon}}$$

for any $\varepsilon \in (0, \varepsilon_0)$.

Proof: Let us prove the first assertion. It suffices to estimate the quantity $P^\varepsilon(B_+^c \cap C_+^\varepsilon)$. But:

$$P^\varepsilon(B_+^c \cap C_+^\varepsilon) \leq P^\varepsilon((B_+ \cup B_-)^c \cap C_\varepsilon) + P^\varepsilon(B_- \cap \{|Z_\varepsilon^o - 2\rho_0(1 + \frac{H_-^2}{3})| \geq \theta\})$$

where $\theta = \frac{\rho_0}{3}(H_+^2 - H_-^2)$. The desired estimate then follows from Proposition 3.1 and Lemma 4.1$\square$

Proof of lemma 4.1: Let us prove the first estimate. We need to estimate the following three events (again we drop the dependence on ε for notational convenience):

$$G = \{|\sum_{i\,\text{odd}} (\alpha_i^2 - E[\alpha_i^2])| > \frac{b-a}{3}\theta\},$$

$$H = \{|\sum_{i\,\text{odd}} \beta_i^2| > \frac{b-a}{3}\theta\},$$

$$J = \{|\sum_{i\,\text{odd}} \alpha_i\beta_i| > \frac{b-a}{6}\theta\}.$$

The existence of $c > 0$ s.t. $P^\varepsilon(G) \leq ce^{-c/\varepsilon}$ follows from lemma 2.2. Note moreover that

$$\frac{1}{b-a} \sum_{i\,\text{odd}} \beta_i^2 \leq \varepsilon (H_+ F_+)^2 \left(\sup_{\tau \in [t_1, t_2]} x_\tau^2 \right).$$

Using Lemma 2.3 and the Markov inequality, we then deduce the existence of $c > 0$ s.t. $P^\varepsilon(H) \le ce^{-c/\varepsilon}$. Note that :

$$J \subset \left\{ \Big(\sup_{\{l \text{ odd}\}} |\alpha_l| \Big) \Big(\sup_{\tau \in [t_1, t_2]} |x_\tau| \Big) > \frac{\theta}{6|H_+F_+|} \right\}$$
$$\subset \left\{ \sup_{\{l \text{ odd}} |\alpha_l| > \frac{\theta \varepsilon^{1/4}}{6|H_+F_+|} \right\} \bigcup \left\{ \sup_{\tau \in [t_1, t_2]} |x_\tau| > \varepsilon^{-1/4} \right\}$$

Using Lemma 2.1 and Lemma 2.3, we deduce :

$$P^\varepsilon(H) \le \frac{c}{\varepsilon} e^{-c/\sqrt{\varepsilon}} + ce^{-c/\sqrt{\varepsilon}}.$$

The result now follows from the three above estimates. □

5. The case of two intervals with $H_+H_- < 0$. Third step.

We want now to show how the decision between $\{x_b > 0\}$ and $\{x_b < 0\}$ can be made from the outputs of the two Kalman filters (KF_+) and (KF_-). For $a < e < d < b$, let us define the test statistic:

$$L_\varepsilon = \int_d^b (H_+x_s^+ - H_-x_s^-)\, dy_s - \frac{1}{2\varepsilon} \int_d^b (|H_+x_s^+|^2 - |H_-x_s^-|^2)\, ds.$$

Using the representation :

$$dy_s = \frac{1}{\varepsilon} \hat{h}_s\, ds + d\nu_s^\varepsilon$$

where $\hat{h}_s = E^\varepsilon(h(x_s)/\mathcal{Y}_s)$ and $\{\nu_t^\varepsilon\}$ -the innovation- is a standard Wiener process. L_ε can be rewritten in two ways :

$$L_\varepsilon = \frac{1}{2\varepsilon} \int_d^b |H_+x_s^+ - H_-x_s^-|^2\, ds + \int_d^b (H_+x_s^+ - H_-x_s^-)\, d\nu_s^\varepsilon +$$
$$+ \frac{1}{\varepsilon} \int_d^b (H_+x_s^+ - H_-x_s^-)(\hat{h}_s - H_+x_s^+)\, ds$$

and also:

$$\text{(5.2)} \qquad L_\varepsilon = -\frac{1}{2\varepsilon} \int_d^b |H_+x_s^+ - H_-x_s^-|^2\, ds + \int_d^b (H_+x_s^+ - H_-x_s^-)\, d\nu_s^\varepsilon$$
$$+ \frac{1}{\varepsilon} \int_d^b (H_+x_s^+ - H_-x_s^-)(\hat{h}_s - H_-x_s^-)\, ds$$

Define $C_+^\varepsilon(a,e)$ and $C_-^\varepsilon(a,e)$ as in section 4, but with the interval $[a,b]$ replaced by $[a,e]$. Define moreover :

$$\tau = \inf\{t; t = a + l\varepsilon; t \ge e; |y_{t+\varepsilon} - y_t| < c\} \wedge b$$

where c is the constant which is used for the definition of the event C_ε.

We want to estimate :

$$E^\varepsilon\left[\int_e^T |\hat{h}_s - H_+ x_s^+|^2\, ds; C_+^\varepsilon(a,e)\right]$$

as well as the same quantity with + replaced by −. We deconpose :

$$\hat{h}_s - H_+ x_s^+ = \hat{h}_s - H_+ \tilde{x}_s^+ + H_+(\tilde{x}_s^+ - x_s^+)$$

where $\tilde{x}_s^+$ is the conditional mean of x_s, given $\mathcal{Y}_s$, in the following filtering problem :

$$(5.3)\qquad \begin{aligned} dx_t &= [f(x_t)1_{\{t\le e\}} + F_+ x_t 1_{\{t>e\}}]dt + dw_t^+;\ x_0 \text{ given} \\ dy_t &= \frac{1}{\varepsilon}[h(x_t)1_{\{t\le e\}} + H_+ x_t 1_{\{t>e\}}]dt + dv_t^{\varepsilon,+};\ y_0 = 0 \end{aligned}$$

Define $\lambda_t = F_+ x_t - f(x_t), \gamma_t = H_+ x_t - h(x_t)$,

$$Z_t = \exp\left(\int_e^{t\vee e} \lambda_s\, dw_s - \frac{1}{2}\int_e^{t\vee e} \lambda_s^2\, ds + \frac{1}{\varepsilon}\int_e^{t\vee e} \gamma_s\, dv_s^\varepsilon - \frac{1}{2\varepsilon^2}\int_e^{t\vee e} \gamma_s^2\, ds\right),\ a \le t \le b.$$

Then $Z_t = \frac{dP^+}{dP^\varepsilon}\big|_{\mathcal{F}_t}$, where P^ε is the initial law on $(\Omega, \mathcal{F})$, and $\{w_t^+\}$, $\{v_t^{\varepsilon,+}\}$ are mutually independent standard Wiener process under P^+.

$$\begin{aligned} dZ_t &= \lambda_t Z_t\, dw_t + \frac{1}{\varepsilon}\gamma_t Z_t\, dv_t^\varepsilon,\ t \ge e;\ Z_e = 1 \\ dy_t &= \frac{1}{\varepsilon} h(x_t)\, dt + dv_t^\varepsilon \end{aligned}$$

It follows from the theory of filtering that :

$$d\hat{Z}_t = \frac{1}{\varepsilon}\hat{Z}_t(h_t^+ - \hat{h}_t + \gamma_t^+)\, dv_t^\varepsilon$$

where $\hat{Z}_t = E^\varepsilon(Z_t/\mathcal{Y}_t), h_t^+ = E^+(h(x_t)/\mathcal{Y}_t), \gamma_t^+ = E^+(\gamma_t/\mathcal{Y}_t)$. Clearly, if we define $\tilde{x}_t^+ = E^+(x_t/\mathcal{F}_t)$,

$$d\hat{Z}_t = \frac{1}{\varepsilon}\hat{Z}_t(H_+\tilde{x}_t^+ - \hat{h}_t)\, dv_t^\varepsilon$$

$$\hat{Z}_t = \exp\left(\frac{1}{\varepsilon}\int_e^{t\vee e} (H_+\tilde{x}_s^+ - \hat{h}_s)\, dv_s^\varepsilon - \frac{1}{2\varepsilon^2}\int_e^{t\vee e} |H_+\tilde{x}_s^+ - \hat{h}_s|^2\, ds\right)$$

But $\hat{Z}_T = E^\varepsilon(Z_T/\mathcal{Y}_T)$. It then follows from Jensen's inequality for conditional expectations and the fact that $C_+^\varepsilon(a,e) \in \mathcal{Y}_e$:

$$\begin{aligned} E^\varepsilon\left(\int_e^T |H_+\tilde{x}_s^+ - \hat{h}_s|^2\, ds; C_+^\varepsilon(a,e)\right) \le & \varepsilon^2 E^\varepsilon\left(\int_e^T |F_+ x_s - f(x_s)|^2\, ds; C_+^\varepsilon(a,e)\right) \\ & + E^\varepsilon\left(\int_e^T |H_+ x_s - h(x_s)|^2\, ds; C_+^\varepsilon(a,e)\right) \end{aligned}$$

It now follows readily from Proposition 4.2 :

Lemma 5.1. *For any $\varepsilon_0 > 0$, there exists k s.t. $\forall \varepsilon \in (0, \varepsilon_0)$,*

$$E^{\varepsilon}\left(\int_e^T |H_+\tilde{x}_s^+ - \hat{h}_s|^2\, ds; C_+^{\varepsilon}(a,e) \cap C_{\varepsilon}(a,b)\right) \leq k e^{-k/\sqrt{\varepsilon}}$$

and the same result holds with + replaced by −□

We need now to estimate the difference $|\tilde{x}_s^+ - x_s^+|, e \leq s \leq b$. Note that $\{x_t^+, t \geq e\}$ and $\{\tilde{x}_t^+, t \geq e\}$ are solutions of the same linear filtering problem with different initial laws, the second one being non gaussian. Let $\mathcal{Y}_t^e = \sigma\{y_s - y_e; e \leq s \leq t\}, t \geq e$. Since $\mathcal{Y}_e$ and $\sigma(x_e) \vee \mathcal{Y}_t^e$ are conditionally independent given $\sigma(x_e)$, for $t \geq e$,

$$\tilde{x}_t^+ = E^+[E^+(x_t/x_e, \mathcal{Y}_t^e)/\mathcal{Y}_t]. \tag{5.4}$$

Define $x_{e,t}^+ = E^+(x_t/x_e, \mathcal{Y}_t^e)$. $x_{e,t}^+$ is the output of a Kalman filter. More precisely, we have:

$$\begin{aligned} dx_{e,t}^+ &= F_+ x_{e,t}^+\, dt + \varepsilon^{-1} R_{e,t} H_+ (dy_t - \varepsilon^{-1} H_+ x_{e,t}^+\, dt), t \geq e; x_{e,e}^+ = x_e \\ \frac{d}{dt} R_{e,t} &= 2F_+ R_{e,t} + 1 - \varepsilon^{-2}(H_+ R_{e,t})^2, t \geq e; R_{e,e} = 0. \end{aligned}$$

Define $K_+(t) = \varepsilon^{-1} R_{e,t} H_+$. It is easily seen that $\exists k$ s.t. $\forall t \geq d$,

$$|K_+ - K_+(t)| \leq k e^{-k/\varepsilon}. \tag{5.5}$$

Moreover,

$$\begin{aligned} d(x_t^+ - x_{e,t}^+) &= (F_+ - \varepsilon^{-1} K_+ H_+)(x_t^+ - x_{e,t}^+)\, dt \\ &\quad + (K_+ - K_+(t))(dy_t - \varepsilon^{-1} H_+ x_{e,t}^+\, dt); t \geq e \\ x_e^+ - x_{e,e}^+ &= x_e^+ - x_e. \end{aligned} \tag{5.6}$$

Since $K_+ H_+ > 0$, it follows from (5.5),(5.6) that there exists k s.t. $\forall \varepsilon > 0$, $\forall t \in [d, b]$,

$$|x_t^+ - x_{e,t}^+| \leq k(1 + |x_e| + |x_e^+|) e^{-k/\varepsilon}.$$

Finally, using (5.4), we obtain that $E^+ \int_d^b |x_t^+ - \tilde{x}_t^+|^2\, dt \leq k e^{-k/\varepsilon}$, $\forall \varepsilon > 0$ and for some k. Since P^{ε} and P^+ coincide on a subset of $B_+(a, b)$,

$$\begin{aligned} P^{\varepsilon}\left(\{\frac{1}{\varepsilon}\int_d^b |\tilde{x}_s^+ - x_s^+|^2\, ds \geq \theta\} \cap C_+^{\varepsilon}(a,b)\right) &\leq P^{\varepsilon}(B_+^c(a,b) \cap C_+(a,b)) \\ &\quad + P^+(\frac{1}{\varepsilon}\int_d^b |x_t^+ - \tilde{x}_t^+|^2\, dt \geq \theta) \end{aligned}$$

It then follows:

Lemma 5.2. *For any $\theta, \varepsilon_0 > 0$, there exists $k > 0$ s.t. $\forall \varepsilon \in (0, \varepsilon_0)$,*

$$P^\varepsilon\left(\{\frac{1}{\varepsilon}\int_d^b |\bar{x}_s^+ - x_s^+|^2\, ds \geq \theta\} \cap C_+^\varepsilon(a,b)\right) \leq ke^{-k/\sqrt{\varepsilon}}$$

From Lemmas 5.1 and 5.2, and the analogues with $+$ replaced by $-$, we deduce:

Lemma 5.3. *For any $\theta, \varepsilon_0 > 0$, there exists $k > 0$ s.t. $\forall \varepsilon \in (0, \varepsilon_0)$,*

$$P^\varepsilon\left(\{\frac{1}{\varepsilon}\int_d^b |\hat{h}_s - H_+ x_s^+|^2\, ds \geq \theta\} \cap C_+^\varepsilon(a,b)\right) \leq ke^{-k/\sqrt{\varepsilon}}$$

and

$$P^\varepsilon\left(\{\frac{1}{\varepsilon}\int_d^b |\hat{h}_s - H_- x_s^-|^2\, ds \geq \theta\} \cap C_-^\varepsilon(a,b)\right) \leq ke^{-k/\sqrt{\varepsilon}}$$

□

Theorem 5.4. $\forall \varepsilon_0 > 0, \exists k > 0$ *s.t. for any $\varepsilon \in (0, \varepsilon_0)$,*

$$P^\varepsilon(\{L_\varepsilon < 0\} \cap C_+^\varepsilon(a,b)) \leq e^{-k/\sqrt{\varepsilon}}$$

and

$$P^\varepsilon(\{L_\varepsilon > 0\} \cap C_-^\varepsilon(a,b)) \leq e^{-k/\sqrt{\varepsilon}}.$$

The proof of the theorem relies on the following Lemma:

Lemma 5.5. *Let $z_t = H_+ x_t^+ - H_- x_t^-; \exists \alpha > 0$ s.t. $\forall \varepsilon_0 > 0, \exists k$ s.t. $\forall \varepsilon \in (0, \varepsilon_0)$,*

$$P^\varepsilon\left(\varepsilon^{-1}\int_\varepsilon^b z_s^2\, ds < \alpha\right) \leq e^{-k/\varepsilon}$$

<u>Proof of Lemma 5.5 (outline)</u>: We have:

$$\begin{aligned} dz_t =& (F_+ - \varepsilon^{-1} H_- K_-) z_t\, dt + (F_+ - F_-) H_- x_t^-\, dt \\ &+ (H_+ K_+ - H_- K_-)[\frac{\bar{h}_t - H_+ x_t^+}{\varepsilon}\, dt + d\nu_t^\varepsilon] \\ dz_t =& (F_+ - \varepsilon^{-1} H_+ K_+) z_t\, dt + (F_+ - F_-) H_- x_t^-\, dt \\ &+ (H_+ K_+ - H_- K_-)[\frac{\bar{h}_t - H_- x_t^-}{\varepsilon}\, dt + d\nu_t^\varepsilon] \end{aligned}$$

It follows from the variation of constants formula that both on $C_+(a,b)$ and on $C_-(a,b)$, z_t is the sum of three terms $z_t = z_t^{(1)} + z_t^{(2)} + z_t^{(3)}$, where $z_t^{(1)}$ is of order $\sqrt{\varepsilon}$, $z_t^{(2)}$ is of order ε and the third one is exponentially small. The first term is the crucial one, which solves:

$$dz_t^{(1)} = (F_+ - \varepsilon^{-1} H_- K_-) z_t^{(1)}\, dt + (H_+ K_+ - H_- K_-)\, d\nu_t^\varepsilon$$

with initial data $z_\epsilon^{(1)}$ having the invariant distribution. By introducing a new time τ such that $\varepsilon\tau = t - e$, the required estimate reduces to a large deviations estimate for the ergodic process in the time scale τ, see e.g. Varadhan [12]□

Proof of Theorem 5.4: Let us prove the first estimate only. We now rewrite L_ϵ for the case $\omega \in C_+(a,b)$ as:

$$L_\epsilon = \frac{1}{4\varepsilon}\int_d^b |H_+x_s^+ - H_-x_s^-|^2\,ds + \int_d^b (H_+x_s^+ - H_-x_s^-)\,d\nu_s^\epsilon$$
$$+ \frac{1}{4\varepsilon}\int_d^b |H_+x_s^+ - H_-x_s^-|^2\,ds + \frac{1}{\varepsilon}\int_d^b (H_+x_s^+ - H_-x_s^-)(\bar{h}_s - H_+X_s^+)\,ds$$

We first show that the sum of the last two terms is nonnegative with very high probability. Indeed, it is bounded below by :

$$X = \frac{1}{8\varepsilon}\int_d^b |H_+x_s^+ - H_-x_s^-|^2\,ds - \frac{2}{\varepsilon}\int_d^b |\hat{h}_s - H_+x_s^+|^2\,ds.$$

For any $\theta > 0$,

$$P^\epsilon(\{X<0\}\bigcap C_+^\epsilon(a,b)) \le P^\epsilon\left(\left\{\frac{2}{\varepsilon}\int_d^b |\hat{h}_s - H_+x_s^+|^2\,ds \ge \theta\right\}\bigcap C_+^\epsilon(a,b)\right)$$
$$+ P^\epsilon\left(\frac{1}{8\varepsilon}\int_d^b |H_+x_s^+ - H_-x_s^-|^2\,ds \le \theta\right)$$

It then follows from lemma 5.3 and 5.5 provided θ is chosen adequately that:

$$P^\epsilon(\{X<0\}\bigcap C_+^\epsilon) \le e^{-k/\sqrt{\epsilon}}, \text{ for some } k \text{ and } \varepsilon \text{ small enough.}$$

Let us now consider the first part of L_ϵ. Let us define $M_t = \varepsilon^{-1/2}\int_e^t z_s\,d\nu_s^\epsilon$. We need to estimate the quantity $P^\epsilon(M_t < -<M>_t/4\sqrt{\varepsilon})$. From Lemma 5.5, it suffices to estimate $P^\epsilon(A)$, where

$$A = \{M_t < -<M>_t/4\sqrt{\varepsilon}\}\bigcap\{<M>_t \ge \alpha\}$$

Using the facts that $E^\epsilon[\exp(\lambda M_t - (\lambda^2/2)<M>_t)] = 1$ and on A, if $\lambda < 0$:

$$\lambda M_t - (\lambda^2/2)<M>_t \ge [-\lambda/4\sqrt{\varepsilon} - \lambda^2/2]<M>_t \ge [-\lambda/4\sqrt{\varepsilon} - \lambda^2/2]\alpha$$

Now chosing λ adequately we have that for any $k > 0$ and ε small enough,

$$P^\epsilon(A) \le e^{-k/\sqrt{\epsilon}}$$

The proof is complete□

Therefore, knowing that we are on C_ϵ, L_ϵ is a good test statistic to decide whether we are on $C_+^\epsilon(a,b)$ or on $C_-^\epsilon(a,b)$, i.e. essentially whether $\{x_t > 0, a \le t \le b\}$ or $\{x_t < 0; a \le t \le b\}$. Note that Lemma 5.3 proves that x_t^+ (resp x_t^-) is then a good estimate of $x_t, d \le t \le b$. It follows moreover from the above that the variance of the conditional law is of the order of ε.

6. Summary of the procedure in the first case, and the general case.

Let us first summarize the procedure in the case studied so far of two intervals with $H_+H_- < 0$.

1- At each time $k\varepsilon$, $k \in \mathbb{N}$, we compute $\varepsilon^{-1}[y_{(k+1)\varepsilon} - y_{k\varepsilon}]$, and check whether or not its absolute value exceeds a given quantity c.

2- As soon as the first test is positive over a certain time-interval, we start running the L_ε-test, possibly in a sequential way .

3- As soon as we have an answer from the L_ε-test, we follow the corresponding Kalman filter (one might continue to run the L_ε-test, in order to correct a possible wrong decision).

Note that before we get any answer from the tests, the estimate of x_t is zero. During a given time interval, this is not a very good estimate. But that situation seems to be inevitable. Indeed, numerical results [3] indicate that, just after x_t has crossed zero, the conditional density has two peaks on both sides of zero, and it takes some time before one of the peaks disappears.

The reason why we do not use the results of section 4 in order to build a test for the choice between $\{x_t > 0\}$ and $\{x_t < 0\}$ is that a test based on the approximation of the quadratic variation of an approximate derivative of y_t would not be very robust. Similarly one might wish to replace the test based on the values of $\varepsilon^{-1}(y_{(k+1)\varepsilon} - y_{k\varepsilon})$ for several consecutive k's by a test using the outputs of the Kalman filters. Indeed, on can show that the difference $h(x_t) - H_+x_t^+$ is always at most of the order of $\sqrt{\varepsilon}$. Unfortunately, due to the presence of the local time term in the expression for $h(x_t)$, we were not able to get a good enough estimate for the probability of error associated to such a test.

Let us now discuss briefly the case where $H_+H_- > 0$, and the general situation. In the case of two intervals, $\mathbb{R}_-$ and $\mathbb{R}_+$, with $H_+H_- > 0$, $h(x)$ is one to one, and clearly no test is needed to decide where is x_t. That decision is in that case obvious from the values of x_t^+ and x_t^-. One might also in this case invoque the result of Picard [8] .

Finally, in the general situation, we have to detect each crossing by x_t of the local maxima or minima of the function h, and choose among the several Kalman filters which one to follow. One can either construct L_ε-type tests only between adjacent intervals, or else betwen any pair of two distinct intervals, depending on the confidence one has in the decisions previously taken. The latter obviously depends on the lengths of the various intervals, as well as the difference between the values of adjacent $|H_i|$'s. Let us finally remark that for the case of two intervals $\mathbb{R}_-$ and $\mathbb{R}_+$, the problem can still be solved if $H_+ = -H_-$, provided $F_+ \neq F_-$. However the technique is different, and we do not present it here.

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FILTRES APPROCHÉS POUR UN PROBLÈME DE FILTRAGE NON LINÉAIRE DISCRÉTISÉ AVEC PETIT BRUIT D'OBSERVATION

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Table des matières

1 Filtrage aproché pour le problème non linéaire discrétisé, avec petit bruit d'observation

1.1 Introduction

On considère le problème suivant :

On a un signal $X_.$ solution de l'EDS

$$dX = b(X)dt + \varepsilon^{\gamma}\sigma(X)dw^1 \, , \; X(0) = \xi \tag{1}$$

et on dispose de l'observation vérifiant

$$dY = h(X)dt + \varepsilon^{1-\gamma}dw^2 \, , \; Y(0) = 0 \tag{2}$$

où w^1, w^2 sont des processus de Wiener standard indépendants et ξ est une v.a. indépendante de w^1 et w^2. Le parametre ε est supposé petit et $0 \leq \gamma < \frac{1}{2}$.

On considère la discrétisation la plus simple de ces équations :

$$X_{k+1} = X_k + b(X_k)\Delta t + \varepsilon^{\gamma}\sigma(X_k)\sqrt{\Delta t}\, w_{k+1} \, , \; X_0 = \xi \tag{3}$$

$$y_k = h(X_k) + \frac{\varepsilon^{1-\gamma}}{\sqrt{\Delta t}}\bar{w}_k \tag{4}$$

où X_k est une approximation de X_{t_k} ($t_k = k\Delta t$), w_k et $\bar{w}_k$ sont des bruits blancs gaussiens standard indépendants et ξ une v.a. indépendante de w_k et $\bar{w}_k$.

Soit $\{\hat{X}_k\}$ le filtre optimal pour le problème discret, i.e.,

$$\hat{X}_k = E[X_k | Y_0^k] \, , \; Y_0^k = \sigma(y_i; i = 0, 1, \cdots, k).$$

Puisque, dans le cas général, la determination de $\hat{X}_k$ presente une grande complexité, on aimerait pouvoir construire une "bonne approximation" de ce filtre répresentant un compromis entre les couts en temps de calcul et la précision des résultats. Soit $\{M_k\}$ une telle approximation, à préciser plus loin, au cours de ce travail. On est interesé par la vitesse de convergence de M_k vers $\hat{X}_k$ quand ε devient "petit". Quand $\Delta t \to 0$ on doit pouvoir approcher le problème de filtrage en temps continu (1 - 2).

1.2 Construction des Filtres Approchés (cas linéaire)

1.2.1 Étude de la vitesse de convergence de l'erreur quadratique moyenne pour quelques filtres proposés

On supposera, dans la suite, que $\gamma = 0$ et on se situe dans le cas *linéaire*.

On considère le système

$$\begin{cases} X_{k+1} &= (1+b\Delta t)X_k + \sigma\sqrt{\Delta t}\, w_{k+1}, \quad X_0 = \xi \\ y_k &= hX_k + \dfrac{\varepsilon}{\sqrt{\Delta t}}\varpi_k, \qquad\qquad y_0 = 0 \end{cases} \tag{5}$$

étant ξ, maintenant, une *v.a.* gaussienne.

On rappelle le fait bien connu de que, dans le cas linéaire, l'estimation optimale est donnée par des équations de dimension finie, les équations du filtre de Kalman :

$$\begin{aligned} \hat{X}_k &= (1+b\Delta t)\hat{X}_{k-1} + \frac{hp_{k|k-1}}{\frac{\varepsilon^2}{\Delta t} + h^2 p_{k|k-1}}(y_k - h(1+b\Delta t)\hat{X}_{k-1}) \\ &= (1+b\Delta t)\hat{X}_{k-1} + \frac{h\Delta t p_{k|k-1}}{\varepsilon^2 + h^2\Delta t p_{k|k-1}}(y_k - h(1+b\Delta t)\hat{X}_{k-1}) . \end{aligned} \tag{6}$$

$$\begin{aligned} p_{k+1|k} &= (1+b\Delta t)^2 p_{k|k-1} + \sigma^2\Delta t - \frac{h^2(1+b\Delta t)^2 p_{k|k-1}^2}{\frac{\varepsilon^2}{\Delta t} + h^2 p_{k|k-1}} \\ &= \frac{(1+b\Delta t)^2\frac{\varepsilon^2}{\Delta t}p_{k|k-1}}{\frac{\varepsilon^2}{\Delta t} + h^2 p_{k|k-1}} + \sigma^2\Delta t \\ &= \frac{(1+b\Delta t)^2\varepsilon^2 p_{k|k-1}}{\varepsilon^2 + h^2\Delta t p_{k|k-1}} + \sigma^2\Delta t \end{aligned} \tag{7}$$

En outre,

$$\begin{aligned} p_k &= p_{k|k-1} - \frac{p_{k|k-1}^2 h^2}{\frac{\varepsilon^2}{\Delta t} + h^2 p_{k|k-1}} \\ &= \frac{\frac{\varepsilon^2}{\Delta t}p_{k|k-1}}{\frac{\varepsilon^2}{\Delta t} + h^2 p_{k|k-1}} \\ &= \frac{\varepsilon^2 p_{k|k-1}}{\varepsilon^2 + h^2\Delta t p_{k|k-1}}, \end{aligned} \tag{8}$$

avec les notations:

$$p_k \triangleq E[(X_k - \hat{X}_k)^2]\,,\ p_{k|k-1} \triangleq E[(X_k - \hat{X}_{k|k-1})^2] \text{ et } \hat{X}_{k|k-1} \triangleq E[X_k|Y_0^{k-1}].$$

Notre but est, ne l'oublions pas, de construire un processus $\{M_k\}$ qui approche $\{\hat{X}_k\}$.

A) On commence par déterminer la covariance de l'erreur de prévision $p_{k|k-1}$ dans une situation stationnaire, ce qui est équivalent à calculer la valeur stationnaire de la covariance de l'erreur d' estimation p_k (notée p_s^+), puisqu'on passe d'une à l'autre par l'expression (8).

Soit p_s la valeur stationnaire de $p_{k|k-1}$.

$$p_s = \frac{(1+b\Delta t)^2\varepsilon^2 p_s}{\varepsilon^2 + h^2\Delta t p_s} + \sigma^2\Delta t$$

$$\varepsilon^2 p_s + h^2\Delta t p_s^2 = (1+b\Delta t)^2\varepsilon^2 p_s + \sigma^2\varepsilon^2\Delta t + \sigma^2 h^2\Delta t^2 p_s$$

$$h^2\Delta t p_s^2 + [\varepsilon^2 - (1+b\Delta t)^2\varepsilon^2 - \sigma^2 h^2\Delta t^2]p_s - \sigma^2\varepsilon^2\Delta t = 0$$

$$p_s = \frac{-[\varepsilon^2 - (1+b\Delta t)^2\varepsilon^2 - \sigma^2h^2\Delta t^2] + r(\varepsilon,\Delta t)}{2h^2\Delta t},$$

$$\begin{aligned} \text{où } r(\varepsilon,\Delta t) &\triangleq \left[[\varepsilon^2 - (1+b\Delta t)^2\varepsilon^2 - \sigma^2h^2\Delta t^2]^2 + 4\sigma^2h^2\varepsilon^2\Delta t^2\right]^{\frac{1}{2}} \\ &= \Delta t\left[[(2b+b^2\Delta t)\varepsilon^2 + \sigma^2h^2\Delta t]^2 + 4\sigma^2h^2\varepsilon^2\right]^{\frac{1}{2}} \\ &\triangleq \Delta t\rho(\varepsilon,\Delta t), \end{aligned}$$

i.e.

$$\begin{aligned} p_s &= \frac{\Delta t[(2b+b^2\Delta t)\varepsilon^2 + \sigma^2h^2\Delta t] + \Delta t\rho(\varepsilon,\Delta t)}{2h^2\Delta t} \\ &= \frac{(2b+b^2\Delta t)\varepsilon^2 + \sigma^2h^2\Delta t + \rho(\varepsilon,\Delta t)}{2h^2} \end{aligned}$$

Le gain stationnaire sera donc,

$$\theta_s = \frac{hp_s\Delta t}{\varepsilon^2 + h^2\Delta t p_s} = \frac{h\Delta t}{\varepsilon^2}p_s^+.$$

On notera θ_k le gain à l'instant t_k :

$$\theta_k \triangleq \frac{h\Delta t p_{k|k-1}}{\varepsilon^2 + h^2\Delta t p_{k|k-1}} = \frac{h\Delta t}{\varepsilon^2}p_k.$$

B) On considère un schéma qui résulte de (6) en remplaçant θ_k par sa valeur stationnaire θ_s ou, d'une façon plus générale, par une approximation de cette valeur. Dans la suite, $\bar\theta$ désignera donc soit θ_s soit une approximation de θ_s, selon le cas explicité.

On considère alors le processus $\{M_k\}$ donné par l'expression

$$M_{k+1} = (1+b\Delta t)M_k + \bar\theta(y_{k+1} - h(1+b\Delta t)M_k). \tag{9}$$

On obtient ainsi,

i)

$$\begin{aligned}
\hat{X}_{k+1} - M_{k+1} &= (1+b\Delta t)\hat{X}_k + \theta_{k+1}(y_{k+1} - h(1+b\Delta t)\hat{X}_k) - (1+b\Delta t)M_k \\
&\quad -\bar{\theta}(y_{k+1} - h(1+b\Delta t)M_k) \\
&= (1+b\Delta t)(\hat{X}_k - M_k) + (\theta_{k+1} - \bar{\theta})y_{k+1} \\
&\quad -h(1+b\Delta t)(\theta_{k+1}\hat{X}_k - \bar{\theta}M_k)
\end{aligned}$$

Étant donné que $\theta_{k+1} = \bar{\theta} + (\theta_{k+1} - \bar{\theta})$,

$$\begin{aligned}
\hat{X}_{k+1} - M_{k+1} &= (1+b\Delta t)(1-h\bar{\theta})(\hat{X}_k - M_k) \\
&\quad +(\theta_{k+1} - \bar{\theta})(y_{k+1} - h(1+b\Delta t)\hat{X}_k),
\end{aligned}$$

où $y_{k+1} - h(1+b\Delta t)\hat{X}_k \triangleq \nu_k$ est l'innovation :

$$E\nu_k^2 = h^2 p_{k+1|k} + \frac{\varepsilon^2}{\Delta t}.$$

Soit $\bar{\eta}_k \triangleq \theta_k - \bar{\theta}$.

$$\begin{aligned}
E[(\hat{X}_{k+1} - M_{k+1})^2] &= (1+b\Delta t)^2(1-h\bar{\theta})^2 E[(\hat{X}_k - M_k)^2] \\
&\quad +\bar{\eta}_{k+1}^2[h^2 p_{k+1|k} + \frac{\varepsilon^2}{\Delta t}],
\end{aligned} \tag{10}$$

et

$$1 - h\bar{\theta} = 1 - h\theta_* + h(\theta_* - \bar{\theta}) = \frac{\varepsilon^2}{\varepsilon^2 + h^2\Delta t p_*} + h(\theta_* - \bar{\theta}).$$

ii)

$$\begin{aligned}
X_{k+1} - M_{k+1} &= (1+b\Delta t)X_k + \sigma\sqrt{\Delta t}w_{k+1} - (1+b\Delta t)M_k \\
&\quad -\bar{\theta}(y_{k+1} - h(1+b\Delta t)M_k) \\
&= (1+b\Delta t)(1-h\bar{\theta})(X_k - M_k) + \sigma\sqrt{\Delta t}w_{k+1} \\
&\quad -\bar{\theta}(h\sigma\sqrt{\Delta t}w_{k+1} + \frac{\varepsilon}{\sqrt{\Delta t}}\bar{w}_{k+1}), \\
&\qquad \text{puisque } y_{k+1} = hX_{k+1} + \frac{\varepsilon}{\sqrt{\Delta t}}\bar{w}_{k+1} \\
&\qquad\qquad = h(1+b\Delta t)X_k + h\sigma\sqrt{\Delta t}w_{k+1} + \frac{\varepsilon}{\sqrt{\Delta t}}\bar{w}_{k+1} \\
&= (1+b\Delta t)(1-h\bar{\theta})(X_k - M_k) + \sigma\sqrt{\Delta t}(1-h\bar{\theta})w_{k+1} \\
&\quad -\frac{\varepsilon}{\sqrt{\Delta t}}\bar{\theta}\bar{w}_{k+1}.
\end{aligned}$$

D'où

$$\begin{aligned}
E[(X_{k+1} - M_{k+1})^2] &= (1+b\Delta t)^2(1-h\bar{\theta})^2 E[(X_k - M_k)^2] + \sigma^2\Delta t(1-h\bar{\theta})^2 \\
&\quad +\frac{\varepsilon^2}{\Delta t}\bar{\theta}^2.
\end{aligned} \tag{11}$$

iii) Soit $\eta_k \triangleq \theta_k - \theta_s$ et $\mu_k \triangleq p_{k|k-1} - p_s$.

$$
\begin{aligned}
\eta_{k+1} &= \theta_{k+1} - \theta_s \\
&= \frac{h\Delta t p_{k+1|k}}{\varepsilon^2 + h^2\Delta t p_{k+1|k}} - \frac{h p_s \Delta t}{\varepsilon^2 + h^2 \Delta t p_s} \\
&= \frac{h\Delta t[\varepsilon^2 p_{k+1|k} + h^2\Delta t p_{k+1|k} p_s - \varepsilon^2 p_s - h^2 \Delta t p_{k+1|k} p_s]}{[\varepsilon^2 + h^2\Delta t p_{k+1|k}][\varepsilon^2 + h^2\Delta t p_s]} \\
&= \frac{h\Delta t \varepsilon^2}{[\varepsilon^2 + h^2\Delta t p_{k+1|k}][\varepsilon^2 + h^2\Delta t p_s]}\mu_{k+1}
\end{aligned}
$$

et

$$
\begin{aligned}
\mu_{k+1} &= p_{k+1|k} - p_s \\
&= \frac{(1+b\Delta t)^2\varepsilon^2 p_{k|k-1}}{\varepsilon^2 + h^2\Delta t p_{k|k-1}} - \frac{(1+b\Delta t)^2\varepsilon^2 p_s}{\varepsilon^2 + h^2\Delta t p_s} \\
&= \frac{\varepsilon^2(1+b\Delta t)^2\varepsilon^2}{[\varepsilon^2 + h^2\Delta t p_{k|k-1}][\varepsilon^2 + h^2\Delta t p_s]}(p_{k|k-1} - p_s) \\
&= \frac{(1+b\Delta t)^2\varepsilon^4}{[\varepsilon^2 + h^2\Delta t p_{k|k-1}][\varepsilon^2 + h^2\Delta t p_s]}\mu_k \\
&= \frac{(1+b\Delta t)^2\varepsilon^4}{[\varepsilon^2 + h^2\Delta t \mu_k + h^2\Delta t p_s][\varepsilon^2 + h^2\Delta t p_s]}\mu_k \\
&= \frac{c_0}{c_1\mu_k + c_2}\mu_k
\end{aligned}
$$

où

$$
\begin{aligned}
c_0 &\triangleq (1+b\Delta t)^2\varepsilon^4 \\
c_1 &\triangleq h^2\Delta t(\varepsilon^2 + h^2\Delta t p_s) \\
c_2 &\triangleq (\varepsilon^2 + h^2\Delta t p_s)^2
\end{aligned}
$$

Donc,

$$
\mu_{k+1} = \frac{c_0}{c_1}\frac{1}{\mu_k + \frac{c_2}{c_1}}\mu_k. \tag{12}
$$

Suivant un raisonnement par récurrence on trouve l'expression:

$$
\mu_k = \frac{c_0^k}{c_1\mu_0 \sum_{i=0}^{k-1} c_2^{k-1-i}c_0^i + c_2^k}\mu_0 = \left(\frac{c_0}{c_2}\right)^k \frac{c_2}{c_1\mu_0 \sum_{i=0}^{k-1}\left(\frac{c_0}{c_2}\right)^i + c_2}\mu_0 \tag{13}
$$

D'autre part,

$$\begin{aligned}\bar{\eta}_k &= (\theta_k - \theta_*) + (\theta_* - \bar{\theta}) \\ &= \eta_k + (\theta_* - \bar{\theta}).\end{aligned}$$

C) Soit $\boxed{\Delta t = \varepsilon^\alpha}$, $\alpha > 0$. Notre but est de faire une discussion de la vitesse de convergence du schéma (9) pour les différentes valeurs de α.

On rappelle que :

$$p_* = \frac{(2b + b^2\Delta t)\varepsilon^2 + \sigma^2 h^2 \Delta t + \rho_\alpha(\varepsilon, \Delta t)}{2h^2}$$

où

$$\rho_\alpha(\varepsilon) \triangleq [[(2b + b^2\Delta t)\varepsilon^2 + \sigma^2 h^2 \Delta t]^2 + 4\sigma^2 h^2 \varepsilon^2]^{\frac{1}{2}}$$

Donc

$$\boxed{p_* = O(\varepsilon^\alpha \vee \varepsilon)}$$

et

$$\boxed{p_* \geq c(\varepsilon^\alpha \vee \varepsilon)}.$$

Alors, en supposant que $\mu_0 > 0$ (i.e. la valeur initial de la covariance, p_0, est une constante indépendante de ε), de (13) vient que :

$$\mu_k \leq (\frac{c_0}{c_2})^k \mu_0,$$

où

$$\begin{aligned}\frac{c_0}{c_2} &= \frac{(1 + b\varepsilon^\alpha)^2 \varepsilon^4}{[\varepsilon^2 + h^2 \varepsilon^\alpha p_*]^2} < 1, \text{ i.e. } c_0 < c_2, \text{ puisque} \\ A &\triangleq 1 - \frac{c_0}{c_2} = \frac{[\varepsilon^2 + h^2\varepsilon^\alpha p_* - (1 + b\varepsilon^\alpha)\varepsilon^2][\varepsilon^2 + h^2\varepsilon^\alpha p_* + (1 + b\varepsilon^\alpha)\varepsilon^2]}{[\varepsilon^2 + h^2\varepsilon^\alpha p_*]^2} \\ &= \frac{\varepsilon^\alpha[h^2 p_* - b\varepsilon^2][\varepsilon^2 + h^2\varepsilon^\alpha p_* + (1 + b\varepsilon^\alpha)\varepsilon^2]}{[\varepsilon^2 + h^2\varepsilon^\alpha p_*]^2} > 0.\end{aligned}$$

D'autre part, les formules (10) et (11) donnent :

-

$$E[(\hat{X}_{k+1} - M_{k+1})^2] = (1-\bar{A})E[(\hat{X}_k - M_k)^2] + B_{k+1} \qquad (14)$$
$$\leq \sum_{i=0}^{k+1}(1-\bar{A})^{k+1-i}B_i \ ,$$

où

$$1-\bar{A} \triangleq (1+b\varepsilon^\alpha)^2(1-h\bar{\theta})^2$$
$$B_{k+1} \triangleq \bar{\eta}^2_{k+1}\frac{\varepsilon^2 + h^2\varepsilon^\alpha p_{k+1|k}}{\varepsilon^\alpha}$$
$$B_0 \triangleq E[(\hat{X}_0 - M_0)^2].$$

-

$$E[(X_{k+1} - M_{k+1})^2] = (1-\bar{A})E[(X_k - M_k)^2] + D$$
$$= (1-\bar{A})^{k+1}E[(X_0 - M_0)^2] + \sum_{i=1}^{k+1}(1-\bar{A})^{k+1-i}D$$
$$= (1-\bar{A})^{k+1}E[(X_0 - M_0)^2] + D\sum_{i=0}^{k}(1-\bar{A})^i$$

Si $\bar{A} > 0$,

$$E[(X_{k+1} - M_{k+1})^2] = (1-\bar{A})^{k+1}E[(X_0 - M_0)^2] + \frac{D}{\bar{A}} \qquad (15)$$

où

$$D \triangleq \sigma^2\varepsilon^\alpha(1-h\bar{\theta})^2 + \frac{\varepsilon^2}{\varepsilon^\alpha}\bar{\theta}^2.$$

I. Supposons d'abord, pour simplifier, qu'on prend comme approximation du gain θ_k le gain stationnaire θ_s, i.e. $\bar{\theta} \equiv \theta_s$ (voir le schèma (9)). On déduira par la suite les résultats pour d'autres approximations $\bar{\theta}$ à préciser.

Dans ce cas là :

i)

$$1-\bar{A} = (1+b\varepsilon^\alpha)^2(1-h\theta_s)^2$$
$$= \frac{(1+b\varepsilon^\alpha)^2\varepsilon^4}{(\varepsilon^2 + h^2\varepsilon^\alpha p_s)^2}$$

$$
\begin{aligned}
&\equiv \frac{c_0}{c_2} \triangleq 1 - A \\
B_{i+1} &= \eta_{i+1}^2 \frac{\varepsilon^2 + h^2\varepsilon^\alpha p_{i+1|i}}{\varepsilon^\alpha} \\
&= \left[\frac{h\varepsilon^{\alpha+2}}{[\varepsilon^2 + h^2\varepsilon^\alpha p_{i+1|i}][\varepsilon^2 + h^2\varepsilon^\alpha p_s]}\mu_{i+1}\right]^2 \frac{\varepsilon^2 + h^2\varepsilon^\alpha p_{i+1|i}}{\varepsilon^\alpha} \\
&= \frac{h^2\varepsilon^{\alpha+4}}{[\varepsilon^2 + h^2\varepsilon^\alpha p_{i+1|i}][\varepsilon^2 + h^2\varepsilon^\alpha p_s]^2}\mu_{i+1}^2
\end{aligned}
$$

et, puisque $p_{i+1|i} \geq p_s$,

$$
B_{i+1} \leq \frac{h^2\varepsilon^{\alpha+4}}{[\varepsilon^2 + h^2\varepsilon^\alpha p_s]^3}\mu_{i+1}^2 = B\mu_{i+1}^2 , \qquad \text{où} \qquad B \triangleq \frac{h^2\varepsilon^{\alpha+4}}{[\varepsilon^2 + h^2\varepsilon^\alpha p_s]^3} .
$$

et

$$
\mu_{i+1}^2 = [(1-A)^2]^{i+1} H_{i+1}\mu_0^2 , \qquad \text{où } H_{i+1} \triangleq \left[\frac{c_2}{c_1\mu_0 \sum_{j=0}^{i}(\frac{c_0}{c_2})^j + c_2}\right]^2
$$

$$
\text{et } H_0 \text{ est tel que } E[(\hat{X}_0 - M_0)^2] = BH_0\mu_0^2.
$$

Reprenant l'expression (14),

$$
\begin{aligned}
E[(\hat{X}_{k+1} - M_{k+1})^2] &\leq (1-A)E[(\hat{X}_k - M_k)^2] + B[(1-A)^2]^{k+1}H_{k+1}\mu_0^2 \\
&\leq \sum_{i=0}^{k+1}(1-A)^{k+1-i}B[(1-A)^2]^i H_i\mu_0^2 \\
&= \mu_0^2 B(1-A)^{k+1}\sum_{i=0}^{k+1}(1-A)^i H_i \qquad (16)
\end{aligned}
$$

Majoration de la série $\left(\sum_{i=0}^{k}(1-A)^i H_i\right)_k$ **:**

$$
\begin{aligned}
H_{i+1} &= \left[\frac{c_2}{c_1\mu_0 \frac{1-(\frac{c_0}{c_2})^{i+1}}{1-\frac{c_0}{c_2}} + c_2}\right]^2 , \qquad \text{avec} \quad \frac{c_0}{c_2} < 1 \\
&= \left[\frac{c_2(1-\frac{c_0}{c_2})}{c_1\mu_0[1-(\frac{c_0}{c_2})^{i+1}] + c_2(1-\frac{c_0}{c_2})}\right]^2
\end{aligned}
$$

$$
\begin{aligned}
&= \left[\frac{c_2(1-\frac{c_0}{c_2})}{c_1\mu_0 + c_2(1-\frac{c_0}{c_2}) - c_1\mu_0(\frac{c_0}{c_2})^{i+1}}\right]^2 \\
&= (c_2-c_0)^2 \frac{1}{(c_1\mu_0)^2[\frac{c_1\mu_0 + c_2 - c_0}{c_1\mu_0} - (\frac{c_0}{c_2})^{i+1}]^2} \\
&= (\frac{c_2-c_0}{c_1\mu_0})^2 \frac{1}{[(1+\frac{c_2-c_0}{c_1\mu_0}) - (\frac{c_0}{c_2})^{i+1}]^2} \\
&= a^2 \frac{1}{(u-r^{i+1})^2}
\end{aligned}
$$

où

$$
\begin{aligned}
a &\triangleq \frac{c_2-c_0}{c_1\mu_0} \\
u &\triangleq 1+\frac{c_2-c_0}{c_1\mu_0} \qquad (17) \\
r &\triangleq \frac{c_0}{c_2} < 1
\end{aligned}
$$

$$(18)$$

Avec ces notations,

$$\sum_{i=1}^{k+1}(1-A)^i H_i = a^2 \sum_{i=1}^{k+1} \frac{r^i}{(u-r^i)^2}$$

On commence par calculer les ordres de grandeur de a et de u.

$$a = \frac{(\varepsilon^2 + h^2\varepsilon^\alpha p_s)^2 - (1+b\varepsilon^\alpha)^2\varepsilon^4}{\mu_0 h^2\varepsilon^\alpha(\varepsilon^2 + h^2\varepsilon^\alpha p_s)}$$

Soit Num le numérateur de a.

$$
\begin{aligned}
Num &\triangleq (\varepsilon^2 + h^2\varepsilon^\alpha p_s)^2 - (1+b\varepsilon^\alpha)^2\varepsilon^4 \\
&= [\varepsilon^2 + h^2\varepsilon^\alpha p_s - (1+b\varepsilon^\alpha)\varepsilon^2][\varepsilon^2 + h^2\varepsilon^\alpha p_s + (1+b\varepsilon^\alpha)\varepsilon^2] \\
&= h^2\varepsilon^\alpha(p_s - \frac{b}{h^2}\varepsilon^2)[\varepsilon^2 + h^2\varepsilon^\alpha p_s + (1+b\varepsilon^\alpha)\varepsilon^2]
\end{aligned}
$$

- Pour $\alpha \geq 1$, $p_s - \frac{b}{h^2}\varepsilon^2 = O(\varepsilon)$, ce qui entraine $Num = O(\varepsilon^{\alpha+3})$ et donc

$$a = O(\varepsilon)$$

Quant à u, on voit rapidement que

$$\begin{aligned} u &= 1 + \frac{(p_s - \frac{b}{h^2}\varepsilon^2)[\varepsilon^2 + h^2\varepsilon^\alpha p_s + (1+b\varepsilon^\alpha)\varepsilon^2]}{\mu_0(\varepsilon^2 + h^2\varepsilon^\alpha p_s)} \\ &= 1 + O(\varepsilon) \text{ et } u > 1. \end{aligned}$$

On peut faire la majoration suivante :

$$\begin{aligned} \sum_{i=1}^{k+1} \frac{r^i}{(u-r^i)^2} &\leq \sum_{i=1}^{\infty} \frac{r^i}{(u-r^i)^2} \\ &\leq \int_0^\infty \frac{r^x}{(u-r^x)^2} dx \\ &= \frac{-1}{\log r} \frac{1}{ua} \end{aligned}$$

Donc

$$\begin{aligned} \sum_{i=0}^{k+1} (1-A)^i H_i &\leq a^2 \frac{-1}{\log r} \frac{1}{ua} + H_0 \\ &= a \frac{-1}{\log r} \frac{1}{u} + H_0 \end{aligned}$$

L'expression (16) vient alors,

$$E[(\hat{X}_{k+1} - M_{k+1})^2] \leq \mu_0^2 B (1-A)^{k+1} (a \frac{-1}{\log r} \frac{1}{u} + H_0) .$$

Etant donné que

$$\begin{aligned} \frac{Ba}{u} &= \frac{\frac{h^2\varepsilon^{\alpha+4}}{[\varepsilon^2 + h^2\varepsilon^\alpha p_s]^3} \frac{(p_s - \frac{b}{h^2}\varepsilon^2)[\varepsilon^2 + h^2\varepsilon^\alpha p_s + (1+b\varepsilon^\alpha)\varepsilon^2]}{\mu_0(\varepsilon^2 + h^2\varepsilon^\alpha p_s)}}{1 + \frac{(p_s - \frac{b}{h^2}\varepsilon^2)[\varepsilon^2 + h^2\varepsilon^\alpha p_s + (1+b\varepsilon^\alpha)\varepsilon^2]}{\mu_0(\varepsilon^2 + h^2\varepsilon^\alpha p_s)}} \\ &= \frac{h^2\varepsilon^{\alpha+4}(p_s - \frac{b}{h^2}\varepsilon^2)[\varepsilon^2 + h^2\varepsilon^\alpha p_s + (1+b\varepsilon^\alpha)\varepsilon^2]}{[\varepsilon^2 + h^2\varepsilon^\alpha p_s]^3[\mu_0(\varepsilon^2 + h^2\varepsilon^\alpha p_s) + (p_s - \frac{b}{h^2}\varepsilon^2)[\varepsilon^2 + h^2\varepsilon^\alpha p_s + (1+b\varepsilon^\alpha)\varepsilon^2]]} \\ &= O(\varepsilon^{\alpha-1}) . \end{aligned}$$

et

$$\frac{-1}{\log r} \leq \frac{1}{1-r} = \frac{1}{A} = O(\varepsilon^{-(\alpha-1)})$$

on obtient :

$$\frac{\frac{Ba}{u}}{-\log r} \leq c \quad \text{et} \quad BH_0 = \frac{1}{\mu_0^2} E[(\hat{X}_0 - M_0)^2]. \tag{19}$$

Donc, si $\alpha \geq 1$,

$$E[(\hat{X}_{k+1} - M_{k+1})^2] \leq c \exp\{-c(k+1)\varepsilon^{\alpha-1}\}$$

et, puisque $t_{k+1} = (k+1)\varepsilon^\alpha$,

$$\boxed{E[(\hat{X}_{k+1} - M_{k+1})^2] \leq c \exp\{-ct_{k+1}\tfrac{1}{\varepsilon}\}}.$$

- Pour $\alpha < 1$, reprenant la formule (14), on obtient:

$$\begin{aligned}
E[(\hat{X}_{k+1} - M_{k+1})^2] &\leq \sum_{i=0}^{k+1} (\frac{c_0}{c_2})^{k+1-i} B\mu_i^2 \\
&\leq B \sum_{i=0}^{k+1} (\frac{c_0}{c_2})^{k+1-i} (\frac{c_0}{c_2})^{2i} \mu_0^2, \text{ puisque } \mu_i \leq (\frac{c_0}{c_2})^i \mu_0 \\
& \qquad \text{car } H_i \leq 1 \\
&= \mu_0^2 (\frac{c_0}{c_2})^{k+1} B \sum_{i=0}^{k+1} (\frac{c_0}{c_2})^i \\
&\leq \mu_0^2 (\frac{c_0}{c_2})^{k+1} \frac{B}{A}
\end{aligned}$$

avec

$$\begin{aligned}
\frac{B}{A} &= \frac{\dfrac{h^2\varepsilon^{\alpha+4}}{[\varepsilon^2 + h^2\varepsilon^\alpha p_s]^3}}{\dfrac{\varepsilon^\alpha[h^2p_s - b\varepsilon^2][\varepsilon^2 + h^2\varepsilon^\alpha p_s + (1+b\varepsilon^\alpha)\varepsilon^2]}{[\varepsilon^2 + h^2\varepsilon^\alpha p_s]^2}} \\
&= \frac{h^2\varepsilon^4}{[\varepsilon^2 + h^2\varepsilon^\alpha p_s][h^2p_s - b\varepsilon^2][\varepsilon^2 + h^2\varepsilon^\alpha p_s + (1+b\varepsilon^\alpha)\varepsilon^2]} \\
&= O(\varepsilon^{4-5\alpha}), \text{ puisque } h^2p_s - b\varepsilon^2 \geq c(\varepsilon^\alpha).
\end{aligned}$$

Donc, si $\alpha < 1$,

$$E[(\hat{X}_{k+1} - M_{k+1})^2] \leq c\varepsilon^{4(k+1)(1-\alpha)}\varepsilon^{4-5\alpha} = c\varepsilon^{4(k+2)(1-\alpha)-\alpha}$$

i.e.

$$\boxed{E[(\hat{X}_{k+1} - M_{k+1})^2] \leq c \exp\{[4(\tfrac{t_{k+1}}{\varepsilon} + 1)(1-\alpha) - \alpha] \log \varepsilon\}}.$$

ii) Quant à l'écart par rapport au signal $\{X_k\}$, puisque

$$\begin{aligned} D &= \sigma^2\varepsilon^\alpha[\frac{\varepsilon^2}{\varepsilon^2+h^2\varepsilon^\alpha p_s}]^2+\frac{\varepsilon^2}{\varepsilon^\alpha}[\frac{h\varepsilon^\alpha p_s}{\varepsilon^2+h^2\varepsilon^\alpha p_s}]^2 \\ &= \frac{\varepsilon^{\alpha+2}}{[\varepsilon^2+h^2\varepsilon^\alpha p_s]^2}[\sigma^2\varepsilon^2+h^2p_s^2]\,, \\ \frac{D}{\bar{A}} &\equiv \frac{D}{A}=\frac{\dfrac{\varepsilon^{\alpha+2}}{[\varepsilon^2+h^2\varepsilon^\alpha p_s]^2}[\sigma^2\varepsilon^2+h^2p_s^2]}{\dfrac{\varepsilon^\alpha[h^2p_s-b\varepsilon^2][\varepsilon^2+h^2\varepsilon^\alpha p_s+(1+b\varepsilon^\alpha)\varepsilon^2]}{[\varepsilon^2+h^2\varepsilon^\alpha p_s]^2}} \\ &= \frac{\varepsilon^2[\sigma^2\varepsilon^2+h^2p_s^2]}{[h^2p_s-b\varepsilon^2][\varepsilon^2+h^2\varepsilon^\alpha p_s+(1+b\varepsilon^\alpha)\varepsilon^2]} \\ &= \begin{cases} O(\varepsilon)\,, & \alpha\geq 1 \\ O(\varepsilon^{2-\alpha})\,, & \alpha\leq 1 \end{cases} \end{aligned}$$

et donc

- si $\alpha \geq 1$,

$$\boxed{E[(X_{k+1}-M_{k+1})^2]\leq \exp\{-ct_{k+1}\tfrac{1}{\varepsilon}\}E[(X_0-M_0)^2]+c\varepsilon} \tag{20}$$

- si $\alpha \leq 1$,

$$\boxed{E[(X_{k+1}-M_{k+1})^2]\leq \exp\{-ct_{k+1}\tfrac{1}{\varepsilon^\alpha}\}E[(X_0-M_0)^2]+c\varepsilon^{2-\alpha}}\ . \tag{21}$$

Remarque 1.1 On peut constater que, quoique soit $\alpha > 0$, les valeurs stationnaires de $E[(X_{k+1}-M_{k+1})^2]$ et $E[(X_{k+1}-\hat{X}_{k+1})^2]$ sont de même ordre de grandeur donc l'utilisation du filtre approché (voir le schèma (9)) est justifiée.

II. On cherche maintenant les expressions plus générales qu'on obtient quand on utilise une approximation $\bar{\theta}$ du gain asymptotique θ_s.

Pour la construction de cette approximation on peut procéder de deux manières:

II.1. On utilise un développement limité de $\rho_\alpha(\varepsilon)$ pour construire une approximation $\bar{p}$ de p_s et on obtient donc une approximation $\bar{\theta}$ de θ_s, par $\bar{\theta} = \dfrac{h\bar{p}\varepsilon^\alpha}{\varepsilon^2+h^2\varepsilon^\alpha\bar{p}}$.

Supposons que $p_s - \bar{p} = O(\varepsilon^m)$, $m > 1$, i.e. $\bar{p}$ est une approximation de p_s d'ordre ε^m.

Alors,

II.1.i)

$$E[(\hat{X}_{k+1} - M_{k+1})^2] \leq \sum_{i=0}^{k+1} (1-\bar{A})^{k+1-i} B_i \,,$$

où, on rappelle,

$$\begin{aligned}
1 - \bar{A} &= \frac{(1+b\varepsilon^\alpha)^2 \varepsilon^4}{[\varepsilon^2 + h^2\varepsilon^\alpha \bar{p}]^2} \\
B_i &= \bar{\eta}_i^2 \frac{\varepsilon^2 + h^2\varepsilon^\alpha p_{i|i-1}}{\varepsilon^\alpha} \\
&= \frac{h^2\varepsilon^{\alpha+4}}{[\varepsilon^2 + h^2\varepsilon^\alpha p_{i|i-1}][\varepsilon^2 + h^2\varepsilon^\alpha \bar{p}]^2} \bar{\mu}_i^2 \\
&\leq \frac{h^2\varepsilon^{\alpha+4}}{[\varepsilon^2 + h^2\varepsilon^\alpha p_s][\varepsilon^2 + h^2\varepsilon^\alpha \bar{p}]^2} \bar{\mu}_i^2 \\
&= \bar{B}\bar{\mu}_i^2 \,, \text{ étant } \bar{B} \triangleq \frac{h^2\varepsilon^{\alpha+4}}{[\varepsilon^2 + h^2\varepsilon^\alpha p_s][\varepsilon^2 + h^2\varepsilon^\alpha \bar{p}]^2} \,.
\end{aligned}$$

Donc

$$\begin{aligned}
E[(\hat{X}_{k+1} - M_{k+1})^2] &\leq \bar{B} \sum_{i=0}^{k+1} (1-\bar{A})^{k+1-i} [\mu_i + c\varepsilon^m]^2 \\
&\leq 2\bar{B} \sum_{i=0}^{k+1} (1-\bar{A})^{k+1-i} \mu_i^2 + 2c\bar{B}\varepsilon^{2m} \sum_{i=0}^{k+1} (1-\bar{A})^{k+1-i} \\
&= 2\bar{B} \sum_{i=0}^{k+1} (1-\bar{A})^{k+1-i} \mu_i^2 + c\bar{B}\varepsilon^{2m} \sum_{i=0}^{k+1} (1-\bar{A})^{i} \\
&\leq 2\bar{B} \sum_{i=0}^{k+1} (1-\bar{A})^{k+1-i} \mu_i^2 + c\varepsilon^{2m} \frac{\bar{B}}{\bar{A}}
\end{aligned}$$

- Pour $\alpha \geq 1$, en utilisant les majorations dans le paragraphe C)I., on obtient:

$$\begin{aligned}
E[(\hat{X}_{k+1} - M_{k+1})^2] &\leq 2\bar{B} \sum_{i=0}^{k+1} (1-\bar{A})^{k+1-i} [(1-A)^2]^i H_i \mu_0^2 + c\varepsilon^{2m} \frac{\bar{B}}{\bar{A}} \\
&= 2\bar{B}\mu_0^2 (1-\bar{A})^{k+1} \sum_{i=0}^{k+1} [\frac{(1-A)^2}{1-\bar{A}}]^i H_i + c\varepsilon^{2m} \frac{\bar{B}}{\bar{A}}
\end{aligned}$$

(a) si $p_s - \bar{p} \geq 0$, alors $1 - \bar{A} \geq 1 - A$ donc

$$E[(\hat{X}_{k+1} - M_{k+1})^2] \leq 2\bar{B}\mu_0^2 (1-\bar{A})^{k+1} \sum_{i=0}^{k+1} (1-A)^i H_i + c\varepsilon^{2m} \frac{\bar{B}}{\bar{A}} \,,$$

$$\leq \quad 2\bar{B}\mu_0^2(1-\bar{A})^{k+1}(a\frac{-1}{\log r}\frac{1}{u}+H_0)+c\varepsilon^{2m}\frac{\bar{B}}{\bar{A}}$$

où a, u et r sont donnés par (17).

(b) si $p_s - \bar{p} \leq 0$, alors $1-\bar{A} \leq 1-A$ donc

$$\begin{aligned} E[(\hat{X}_{k+1}-M_{k+1})^2] &\leq 2\bar{B}\mu_0^2(1-A)^{k+1}\sum_{i=0}^{k+1}(1-A)^i H_i + c\varepsilon^{2m}\frac{\bar{B}}{\bar{A}} , \\ &\leq 2\bar{B}\mu_0^2(1-A)^{k+1}(a\frac{-1}{\log r}\frac{1}{u}+H_0)+c\varepsilon^{2m}\frac{\bar{B}}{\bar{A}} . \end{aligned}$$

- Pour $\alpha < 1$,

$$\begin{aligned} E[(\hat{X}_{k+1}-M_{k+1})^2] &\leq 2\mu_0^2\bar{B}(1-\bar{A})^{k+1}\sum_{i=0}^{k+1}[\frac{(1-A)^2}{1-\bar{A}}]^i + c\varepsilon^{2m}\frac{\bar{B}}{\bar{A}} \\ &\leq 2\mu_0^2(1-\bar{A})^{k+1}\frac{\bar{B}}{1-\frac{(1-A)^2}{1-\bar{A}}} + c\varepsilon^{2m}\frac{\bar{B}}{\bar{A}} \end{aligned}$$

Quelques calculs rapides nous donnent:

$$\begin{aligned} \bar{B}\frac{a}{u} &= \frac{h^2\varepsilon^{\alpha+4}}{[\varepsilon^2+h^2\varepsilon^\alpha p_s][\varepsilon^2+h^2\varepsilon^\alpha\bar{p}]^2} \\ &\cdot\frac{(p_s-\frac{b}{h^2}\varepsilon^2)[\varepsilon^2+h^2\varepsilon^\alpha p_s+(1+b\varepsilon^\alpha)\varepsilon^2]}{\mu_0(\varepsilon^2+h^2\varepsilon^\alpha p_s)+(p_s-\frac{b}{h^2}\varepsilon^2)[\varepsilon^2+h^2\varepsilon^\alpha p_s+(1+b\varepsilon^\alpha)\varepsilon^2]} \\ &= O(\varepsilon^{\alpha-1}) \text{ , si } \alpha \geq 1 \end{aligned}$$

$$\begin{aligned} \frac{\bar{B}}{1-\frac{(1-A)^2}{1-\bar{A}}} &= \frac{\bar{B}(1-\bar{A})}{(1-\bar{A})-(1-A)^2} \\ &= \frac{\frac{h^2\varepsilon^{\alpha+4}}{[\varepsilon^2+h^2\varepsilon^\alpha p_s][\varepsilon^2+h^2\varepsilon^\alpha\bar{p}]^2}\frac{(1+b\varepsilon^\alpha)^2\varepsilon^4}{[\varepsilon^2+h^2\varepsilon^\alpha\bar{p}]^2}}{\frac{(1+b\varepsilon^\alpha)^2\varepsilon^4}{[\varepsilon^2+h^2\varepsilon^\alpha\bar{p}]^2}-\frac{(1+b\varepsilon^\alpha)^4\varepsilon^8}{[\varepsilon^2+h^2\varepsilon^\alpha p_s]^4}} \\ &= \frac{\frac{h^2\varepsilon^{\alpha+4}}{[\varepsilon^2+h^2\varepsilon^\alpha p_s][\varepsilon^2+h^2\varepsilon^\alpha\bar{p}]^2}}{\frac{[\varepsilon^2+h^2\varepsilon^\alpha p_s]^4-(1+b\varepsilon^\alpha)^2\varepsilon^4[\varepsilon^2+h^2\varepsilon^\alpha\bar{p}]^2}{[\varepsilon^2+h^2\varepsilon^\alpha p_s]^4}} \end{aligned}$$

$$
\begin{aligned}
&= \frac{h^2\varepsilon^{\alpha+4}}{[\varepsilon^2+h^2\varepsilon^\alpha \bar{p}]^2[(\varepsilon^2+h^2\varepsilon^\alpha p_s)^2+\varepsilon^2(1+b\varepsilon^\alpha)(\varepsilon^2+h^2\varepsilon^\alpha\bar{p})]} \\
&\quad\cdot \frac{[\varepsilon^2+h^2\varepsilon^\alpha p_s]^3}{(\varepsilon^2+h^2\varepsilon^\alpha p_s)^2-\varepsilon^2(1+b\varepsilon^\alpha)(\varepsilon^2+h^2\varepsilon^\alpha\bar{p})}\,, \\
&\quad \text{où} \qquad (\varepsilon^2+h^2\varepsilon^\alpha p_s)^2-\varepsilon^2(1+b\varepsilon^\alpha)(\varepsilon^2+h^2\varepsilon^\alpha\bar{p}) \\
&\qquad\qquad = h^2\varepsilon^{\alpha+2}(2p_s-\bar{p})+h^4\varepsilon^{2\alpha}p_s^2-b\varepsilon^{\alpha+4}-h^2\varepsilon^{2\alpha+2}\bar{p} \\
&\qquad\qquad \geq c\varepsilon^{4\alpha} \\
&= O(\varepsilon^{4-5\alpha})\,,\ \text{si } \alpha<1\,.
\end{aligned}
$$

$$
\begin{aligned}
\bar{A} &= 1-\frac{(1+b\varepsilon^\alpha)^2\varepsilon^4}{[\varepsilon^2+h^2\varepsilon^\alpha\bar{p}]^2} \\
&= \frac{[\varepsilon^2+h^2\varepsilon^\alpha\bar{p}]^2-(1+b\varepsilon^\alpha)^2\varepsilon^4}{[\varepsilon^2+h^2\varepsilon^\alpha\bar{p}]^2} \\
&= \frac{[\varepsilon^2+h^2\varepsilon^\alpha\bar{p}-(1+b\varepsilon^\alpha)\varepsilon^2][\varepsilon^2+h^2\varepsilon^\alpha\bar{p}+(1+b\varepsilon^\alpha)\varepsilon^2]}{[\varepsilon^2+h^2\varepsilon^\alpha\bar{p}]^2} \\
&= \frac{\varepsilon^\alpha[h^2\bar{p}-b\varepsilon^2][\varepsilon^2+h^2\varepsilon^\alpha\bar{p}+(1+b\varepsilon^\alpha)\varepsilon^2]}{[\varepsilon^2+h^2\varepsilon^\alpha\bar{p}]^2}\,, \\
&\quad \text{où } h^2\bar{p}-b\varepsilon^2 \leq c(\varepsilon^\alpha\vee\varepsilon) \\
&= \begin{cases} O(\varepsilon^{\alpha-1}), & \alpha\geq 1 \\ O(1), & \alpha<1 \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\bar{B}}{\bar{A}} &= \frac{\dfrac{h^2\varepsilon^{\alpha+4}}{[\varepsilon^2+h^2\varepsilon^\alpha p_s][\varepsilon^2+h^2\varepsilon^\alpha\bar{p}]^2}}{\dfrac{\varepsilon^\alpha[h^2\bar{p}-b\varepsilon^2][\varepsilon^2+h^2\varepsilon^\alpha\bar{p}+(1+b\varepsilon^\alpha)\varepsilon^2]}{[\varepsilon^2+h^2\varepsilon^\alpha\bar{p}]^2}} \\
&= \frac{h^2\varepsilon^4}{[\varepsilon^2+h^2\varepsilon^\alpha p_s][h^2\bar{p}-b\varepsilon^2][\varepsilon^2+h^2\varepsilon^\alpha\bar{p}+(1+b\varepsilon^\alpha)\varepsilon^2]} \\
&= \begin{cases} O(\frac{1}{\varepsilon}), & \alpha\geq 1 \\ O(\varepsilon^{4-5\alpha}), & \alpha<1 \end{cases}
\end{aligned}
$$

On obtient donc les estimations suivantes:

- **si** $\alpha \geq 1$,

$$
\boxed{E[(\hat{X}_{k+1}-M_{k+1})^2] \leq c\exp\{-ct_{k+1}\tfrac{1}{\varepsilon}\}+c\varepsilon^{2m-1}} \tag{22}
$$

- si $\alpha < 1$, puisque $1 - \bar{A} \leq O(\varepsilon^{4(1-\alpha)})$,

$$\boxed{E[(\hat{X}_{k+1} - M_{k+1})^2] \leq c\exp\{[4(\tfrac{t_{k+1}}{\varepsilon} + 1)(1-\alpha) - \alpha]\log\varepsilon\} + c\varepsilon^{2m+4-5\alpha}} \tag{23}$$

II.1.ii) D'autre part,

$$E[(X_{k+1} - M_{k+1})^2] \leq (1 - \bar{A})^{k+1}E[(X_0 - M_0)^2] + \frac{D}{\bar{A}}$$

où

$$\begin{aligned}
\frac{D}{\bar{A}} &= \frac{\dfrac{\varepsilon^{\alpha+2}[\sigma^2\varepsilon^2 + h^2\bar{p}^2]}{[\varepsilon^2 + h^2\varepsilon^\alpha\bar{p}]^2}}{\dfrac{\varepsilon^\alpha[h^2\bar{p} - b\varepsilon^2][\varepsilon^2 + h^2\varepsilon^\alpha\bar{p} + (1 + b\varepsilon^\alpha)\varepsilon^2]}{[\varepsilon^2 + h^2\varepsilon^\alpha\bar{p}]^2}} \\
&= \frac{\varepsilon^2[\sigma^2\varepsilon^2 + h^2\bar{p}^2]}{[h^2\bar{p} - b\varepsilon^2][\varepsilon^2 + h^2\varepsilon^\alpha\bar{p} + (1 + b\varepsilon^\alpha)\varepsilon^2]} \\
&= \begin{cases} O(\varepsilon)\,, & \alpha \geq 1 \\ O(\varepsilon^{2-\alpha})\,, & \alpha \leq 1 \end{cases},
\end{aligned}$$

donc

- si $\alpha \geq 1$,

$$\boxed{E[(X_{k+1} - M_{k+1})^2] \leq \exp\{-ct_{k+1}\tfrac{1}{\varepsilon}\}E[(X_0 - M_0)^2] + c\varepsilon}$$

- si $\alpha \leq 1$,

$$\boxed{E[(X_{k+1} - M_{k+1})^2] \leq \exp\{-ct_{k+1}\tfrac{1}{\varepsilon^\alpha}\}E[(X_0 - M_0)^2] + c\varepsilon^{2-\alpha}}\,.$$

C'est en fait le même résultat qui a été obtenu dans le cas de l'utilisation du gain stationnaire.

On explicitera maintenant quelques expressions possibles pour l'approximation du gain, $\bar{\theta}$ (ou de la variance, $\bar{p}$):

Supposons que $h > 0$ et $\sigma > 0$.

(a) si $\alpha \geq 2$, le développement de Taylor de $\rho_\alpha(\varepsilon)$ nous donne, après quelques calculs,

$$\rho_\alpha(\varepsilon) = 2h\sigma\varepsilon + O(\varepsilon^3)$$

et donc

$$p_s = \frac{\sigma}{h}\varepsilon + \frac{b}{h^2}\varepsilon^2 + \frac{\sigma^2}{2}\varepsilon^\alpha + O(\varepsilon^3) \, .$$

Si on prend, par exemple,

1. $\boxed{\bar{p} = \frac{\sigma}{h}\varepsilon}$, i.e. $p_s - \bar{p} = O(\varepsilon^2)$, alors

$$E[(\hat{X}_k - M_k)^2] \leq c\exp\{-ct_k\frac{1}{\varepsilon}\} + c\varepsilon^3 \, . \tag{24}$$

2. $\boxed{\bar{p} = \frac{\sigma}{h}\varepsilon + \frac{b}{h^2}\varepsilon^2}$ si $\alpha \neq 2$,i.e. $p_s - \bar{p} = O(\varepsilon^\alpha \vee \varepsilon^3)$, alors

$$E[(\hat{X}_k - M_k)^2] \leq c\exp\{-ct_k\frac{1}{\varepsilon}\} + c(\varepsilon^{2\alpha-1} \vee \varepsilon^5) \, .$$

3. $\boxed{\bar{p} = \frac{\sigma}{h}\varepsilon + \frac{b}{h^2}\varepsilon^2 + \frac{\sigma^2}{2}\varepsilon^\alpha}$ si $2 \leq \alpha < 3$,i.e. $p_s - \bar{p} = O(\varepsilon^3)$, alors

$$E[(\hat{X}_k - M_k)^2] \leq c\exp\{-ct_k\frac{1}{\varepsilon}\} + c\varepsilon^5 \, .$$

(b) si $1 < \alpha \leq 2$, du développement limité de $\rho_\alpha(\varepsilon)$ résulte:

$$p_s = \frac{\sigma}{h}\varepsilon + \frac{\sigma^2}{2}\varepsilon^\alpha + O(\varepsilon^2 \vee \varepsilon^{2\alpha-1}) \tag{25}$$

donc on peut prendre comme approximation de p_s par exemple:

1. $\boxed{\bar{p} = \frac{\sigma}{h}\varepsilon}$, i.e., $p_s - \bar{p} = O(\varepsilon^\alpha)$ et alors

$$E[(\hat{X}_k - M_k)^2] \leq c\exp\{-ct_k\frac{1}{\varepsilon}\} + c\varepsilon^{2\alpha-1}$$

2. $\boxed{\bar{p} = \frac{\sigma}{h}\varepsilon + \frac{\sigma^2}{2}\varepsilon^\alpha}$, i.e., $p_s - \bar{p} = O(\varepsilon^2 \vee \varepsilon^{2\alpha-1})$ et alors

$$E[(\hat{X}_k - M_k)^2] \leq c\exp\{-ct_k\frac{1}{\varepsilon}\} + c(\varepsilon^3 \vee \varepsilon^{4\alpha-3})$$

(c) **si $\alpha = 1$, le développement limité de $\rho_\alpha(\varepsilon)$ est un peu particulier:**

$$\rho_\alpha(\varepsilon) = \sigma h\sqrt{4+\sigma^2h^2}\varepsilon + \frac{2b\sigma h}{\sqrt{4+\sigma^2h^2}}\varepsilon^2 + O(\varepsilon^3)\ ,$$

ce qui entraîne:

$$p_\varepsilon \;=\; \left(\frac{\sigma}{h}\sqrt{1+\frac{\sigma^2h^2}{4}} + \frac{\sigma^2}{2}\right)\varepsilon + \left(\frac{b}{h^2} + \frac{b\sigma}{2h\sqrt{1+\frac{\sigma^2h^2}{4}}}\right)\varepsilon^2 + O(\varepsilon^3)\ .$$

On propose, par exemple, les approximations suivantes:

1. $\boxed{\bar{p} = (\frac{\sigma}{h}\sqrt{1+\frac{\sigma^2h^2}{4}} + \frac{\sigma^2}{2})\varepsilon}$, **i.e. $p_\varepsilon - \bar{p} = O(\varepsilon^2)$ et alors**

$$E[(\hat{X}_k - M_k)^2] \le c\exp\{-ct_k\frac{1}{\varepsilon}\} + c\varepsilon^3$$

2. $\boxed{\bar{p} = (\frac{\sigma}{h}\sqrt{1+\frac{\sigma^2h^2}{4}} + \frac{\sigma^2}{2})\varepsilon + (\frac{b}{h^2} + \frac{b\sigma}{2h\sqrt{1+\frac{\sigma^2h^2}{4}}})\varepsilon^2}$, **i.e. $p_\varepsilon - \bar{p} = O(\varepsilon^3)$ et alors**

$$E[(\hat{X}_k - M_k)^2] \le c\exp\{-ct_k\frac{1}{\varepsilon}\} + c\varepsilon^5\ .$$

(d) **si $0 < \alpha < 1$, le développement de la fonction $\rho_\alpha(\varepsilon)$ devient**

$$\rho_\alpha(\varepsilon) = \sigma^2h^2\varepsilon^\alpha + O(\varepsilon^{2-\alpha})$$

d'où

$$p_\varepsilon = \sigma^2\varepsilon^\alpha + O(\varepsilon^{2-\alpha})\ .$$

Donc, si on prend $\boxed{\bar{p} = \sigma^2\varepsilon^\alpha}$, i.e. $p_\varepsilon - \bar{p} = O(\varepsilon^{2-\alpha})$, alors

$$E[(\hat{X}_k - M_k)^2] \le c\exp\{[4(\frac{t_{k+1}}{\varepsilon} + 1)(1-\alpha) - \alpha]\log\varepsilon]\} + c\varepsilon^{8-7\alpha}\ .$$

II.2. On essaie maintenant de décrire une approximation de θ_s sans passer par le développement de $\rho_\alpha(\varepsilon)$.

$$
\begin{aligned}
\theta_s &= \frac{h\varepsilon^\alpha \dfrac{(2b+b^2\varepsilon^\alpha)\varepsilon^2+\sigma^2h^2\varepsilon^\alpha+\rho_\alpha(\varepsilon)}{2h^2}}{\varepsilon^2+h^2\varepsilon^\alpha\dfrac{(2b+b^2\varepsilon^\alpha)\varepsilon^2+\sigma^2h^2\varepsilon^\alpha+\rho_\alpha(\varepsilon)}{2h^2}} \\
&= \frac{\varepsilon^\alpha[(2b+b^2\varepsilon^\alpha)\varepsilon^2+\sigma^2h^2\varepsilon^\alpha+\rho_\alpha(\varepsilon)]}{h[2\varepsilon^2+\varepsilon^\alpha[(2b+b^2\varepsilon^\alpha)\varepsilon^2+\sigma^2h^2\varepsilon^\alpha+\rho_\alpha(\varepsilon)]]} \\
&= \frac{1}{h}\frac{1}{1+\dfrac{2\varepsilon^2}{\varepsilon^\alpha[(2b+b^2\varepsilon^\alpha)\varepsilon^2+\sigma^2h^2\varepsilon^\alpha+\rho_\alpha(\varepsilon)]}} \\
&= \frac{1}{h}\frac{1}{1+D_0}
\end{aligned}
$$

$$
\begin{aligned}
\text{où} \quad D_0 &\triangleq \frac{2\varepsilon^2}{\varepsilon^\alpha[(2b+b^2\varepsilon^\alpha)\varepsilon^2+\sigma^2h^2\varepsilon^\alpha+\rho_\alpha(\varepsilon)]} \\
&= \frac{2\varepsilon^{2-\alpha}}{(2b+b^2\varepsilon^\alpha)\varepsilon^2+\sigma^2h^2\varepsilon^\alpha+\rho_\alpha(\varepsilon)}
\end{aligned}
$$

$$
\text{et} \quad \rho_\alpha(\varepsilon) \geq c(\varepsilon^\alpha \vee \varepsilon)\,.
$$

II.2.1 Si $\alpha > 1$,

$$
D_0 \approx 2\frac{\varepsilon^2}{\varepsilon^\alpha 2h\sigma\varepsilon} = \frac{1}{h\sigma\varepsilon^{\alpha-1}}\,,
$$

puisque $\rho_\alpha(\varepsilon) = 2h\sigma\varepsilon + O(\varepsilon^3 \vee \varepsilon^{2\alpha-1})$.
Une approximation de θ_s est, par exemple,

$$
\begin{aligned}
\theta^a &= \frac{1}{h}\frac{1}{1+\dfrac{1}{h\sigma\varepsilon^{\alpha-1}}} \\
&\approx \frac{1}{h}\frac{1}{\dfrac{1}{h\sigma\varepsilon^{\alpha-1}}} \\
&= \sigma\varepsilon^{\alpha-1}\,.
\end{aligned}
$$

On prend donc

$$
\boxed{\bar{\theta} = \sigma\varepsilon^{\alpha-1}}\,.
$$

Puisque

$$\bar{\theta} = \frac{h\bar{p}\varepsilon^\alpha}{\varepsilon^2 + h^2\varepsilon^\alpha \bar{p}},$$

i.e., pour $\bar{\theta} \neq \frac{1}{h}$,

$$\bar{p} = \frac{\bar{\theta}\varepsilon^2}{h\varepsilon^\alpha(1-h\bar{\theta})} = \frac{\bar{\theta}}{h\varepsilon^{\alpha-2}(1-h\bar{\theta})},$$

ce qui donne, dans notre cas,

$$\bar{p} = \frac{\sigma\varepsilon^{\alpha-1}}{h\varepsilon^{\alpha-2}(1-h\sigma\varepsilon^{\alpha-1})} = \frac{\sigma\varepsilon}{h(1-\sigma h\varepsilon^{\alpha-1})},$$

on obtient, en utilisant le développement limité de $\dfrac{1}{1-\sigma h\varepsilon^{\alpha-1}}$,

$$\bar{p} = \frac{\sigma}{h}\varepsilon + \sigma^2\varepsilon^\alpha + \sigma^3 h\varepsilon^{2\alpha-1} + O(\varepsilon^{3\alpha-2})$$

d'où

$$p_\varepsilon - \bar{p} = c(\varepsilon^2 \vee \varepsilon^\alpha) \qquad \text{et, si } \alpha > 2\,, \quad p_\varepsilon - \bar{p} > 0\,.$$

Donc,

i) de la formule (22), on déduit les estimations suivantes:

- pour $1 < \alpha \leq 2$,

$$\boxed{E[(\hat{X}_{k+1} - M_{k+1})^2] \leq c\exp\{-ct_{k+1}\tfrac{1}{\varepsilon}\} + c\varepsilon^{2\alpha-1}} \tag{26}$$

- pour $\alpha \geq 2$,

$$\boxed{E[(\hat{X}_{k+1} - M_{k+1})^2] \leq c\exp\{-ct_{k+1}\tfrac{1}{\varepsilon}\} + c\varepsilon^3}\,. \tag{27}$$

ii) de la formule (20),on obtient:

$$\boxed{E[(X_{k+1} - M_{k+1})^2] \leq \exp\{-ct_{k+1}\tfrac{1}{\varepsilon^\alpha}\}E[(X_0 - M_0)^2] + c\varepsilon}\,. \tag{28}$$

Remarque 1.2 Une fois encore on trouve des estimations de $E[(\hat{X}_{k+1} - M_{k+1})^2]$ d'ordre inferieur ou égal à celui de $E[(X_{k+1} - M_{k+1})^2]$.

Remarque 1.3 Le schèma correspondant aux approximations (24) et (25) (i.e. qui utilise le gain $\bar{\theta} = \dfrac{\sigma\varepsilon^{\alpha+1}}{\varepsilon^2 + \sigma h\varepsilon^{\alpha+1}}$) n'a pas d'interêt pratique, puisque son utilisation oblige a un calcul plus compliqué que celui du schéma qu'on vient d'obtenir alors que l'ordre de l'erreur associée reste le même.

II.2.2. Si $\alpha < 1$,

$$D_0 \approx \frac{2\varepsilon^2}{\sigma^2 h^2 \varepsilon^{2\alpha}} = \frac{2}{h^2\sigma^2}\varepsilon^{2(1-\alpha)} .$$

Donc une approximation du gain stationnaire θ_s est, par exemple,

$$\theta^a = \frac{1}{h}\frac{1}{1+\dfrac{2}{h^2\sigma^2}\varepsilon^{2(1-\alpha)}}$$

et on peut considérer

$$\boxed{\bar{\theta} = \tfrac{1}{h}}.$$

Le schéma (9) est alors

$$\begin{aligned} M_{k+1} &= (1+b\varepsilon^\alpha)M_k + \frac{1}{h}(y_{k+1} - h(1+b\varepsilon^\alpha)\hat{X}_k) \\ &= \frac{1}{h}y_{k+1} \end{aligned} \tag{29}$$

Les formules (10) et (11) deviennent:

•

$$\hat{X}_{k+1} - M_{k+1} = -\frac{\varepsilon^2}{h[\varepsilon^2 + h^2\varepsilon^\alpha p_{k+1|k}]}(y_{k+1} - h(1+b\varepsilon^\alpha)M_k)$$

donc

$$E[(\hat{X}_{k+1} - M_{k+1})^2] = -\frac{\varepsilon^4}{h^2\varepsilon^\alpha[\varepsilon^2 + h^2\varepsilon^\alpha p_{k+1|k}]}$$

i.e.

$$\boxed{E[(\hat{X}_{k+1} - M_{k+1})^2] = O\left(\varepsilon^{4-3\alpha}\right)}. \tag{30}$$

• D'autre part,

$$\begin{aligned} X_{k+1} - M_{k+1} &= \frac{y_{k+1} - \frac{\varepsilon}{\varepsilon^{\frac{\alpha}{2}}}w^2_{k+1}}{h} - \frac{y_{k+1}}{h} \\ &= -\frac{\varepsilon}{h\varepsilon^{\frac{\alpha}{2}}}w^2_{k+1} . \end{aligned}$$

Donc

$$\boxed{E[(X_{k+1} - M_{k+1})^2] = \tfrac{1}{h^2}\varepsilon^{2-\alpha}}. \tag{31}$$

II.2.3. Si $\alpha = 1$,

$$\begin{aligned} D_0 &= \frac{2\varepsilon^2}{(2b + b^2\varepsilon^\alpha)\varepsilon^3 + \sigma^2 h^2 \varepsilon^2 + \sigma h\sqrt{4+\sigma^2 h^2}\,\varepsilon + O(\varepsilon^2)} \\ &\approx \frac{2}{\sigma h\sqrt{4+\sigma^2 h^2}}\varepsilon\,. \end{aligned}$$

Quand $\varepsilon \to 0$, une approximation de θ_s est, par exemple,

$$\theta^a = \frac{1}{h}\frac{1}{1 + \dfrac{2}{\sigma h\sqrt{4+\sigma^2 h^2}}\varepsilon}$$

ou, plus simplement,

$$\boxed{\bar{\theta} = \tfrac{1}{h}\,.}$$

On retrouve le schéma (9) et les formules (30) et (31).

1.2.2 Conclusion

On établit les tableaux suivants, lesquels nous donnent les vitesses de convergence à 0 de l'erreur quadratique moyenne pour les différents filtres approchés qui ont fait l'objet de cet étude, mettant en évidence la dépendance de ces filtres selon le rapport variance du bruit d'observation versus pas de temps.

- Pour $\alpha > 1$

gain du filtre approché	*estimation de l'erreur* $E[(\bar{X}_{k+1} - M_{k+1})^2]$
θ_s (gain stationnaire)	$ce^{-c\frac{t_{k+1}}{\varepsilon}}$
$\sigma\varepsilon^{\alpha-1}$	$c(\varepsilon^{2\alpha-1} \vee \varepsilon^3)$
$1 < \alpha \leq 2$ $\quad \frac{\sigma\varepsilon^{\alpha+1}+\frac{\sigma^2 h}{2}\varepsilon^{2\alpha}}{\varepsilon^2+\sigma h\varepsilon^{\alpha+1}+\frac{\sigma^2 h^2}{2}\varepsilon^{2\alpha}}$	$c(\varepsilon^3 \vee \varepsilon^{4\alpha-3})$
$\alpha > 2$ $\quad \frac{\sigma\varepsilon^{\alpha+1}+\frac{b}{h}\varepsilon^{\alpha+2}}{\varepsilon^2+\sigma h\varepsilon^{\alpha+1}+b\varepsilon^{\alpha+2}}$	$c(\varepsilon^{2\alpha-1} \vee \varepsilon^5)$
$2 \leq \alpha < 3$ $\quad \frac{\sigma\varepsilon^{\alpha+1}+\frac{b}{h}\varepsilon^{\alpha+2}+\frac{\sigma^2 h}{2}\varepsilon^{2\alpha}}{\varepsilon^2+\sigma h\varepsilon^{\alpha+1}+b\varepsilon^{\alpha+2}+\frac{\sigma^2 h^2}{2}\varepsilon^{2\alpha}}$	$c\varepsilon^5$

D'autre part, on a:

$$E[(X_{k+1} - M_{k+1})^2] \leq ce^{-c\frac{t_{k+1}}{\varepsilon}} + c\varepsilon \ .$$

- Pour $\alpha = 1$

gain du filtre approché	*estimation de l'erreur* $E[(\bar{X}_{k+1} - M_{k+1})^2]$
θ_s (gain stationnaire)	$ce^{-c\frac{t_{k+1}}{\varepsilon}}$
$\frac{1}{h}$	$c\varepsilon$
$\frac{\sigma\sqrt{1+\frac{\sigma^2 h^2}{4}}+\frac{\sigma^2 h}{2}}{1+h(\sigma\sqrt{1+\frac{\sigma^2 h^2}{4}}+\frac{\sigma^2 h}{2})}$	$c\varepsilon^3$
$\frac{\sigma\sqrt{1+\frac{\sigma^2 h^2}{4}}+\frac{\sigma^2 h}{2}+(\frac{b}{h}+\frac{b\sigma}{\sqrt{1+\frac{\sigma^2 h^2}{4}}})\varepsilon}{1+h(\sigma\sqrt{1+\frac{\sigma^2 h^2}{4}}+\frac{\sigma^2 h}{2})+(b+\frac{b\sigma h}{2\sqrt{1+\frac{\sigma^2 h^2}{4}}})\varepsilon}$	$c\varepsilon^5$

D'autre part, on a:

$$E[(X_{k+1} - M_{k+1})^2] \leq ce^{-c\frac{t_{k+1}}{\varepsilon}} + c\varepsilon \ .$$

- Pour $\alpha < 1$

gain du filtre approché	*estimation de l'erreur*
	$E[(\tilde{X}_{k+1} - M_{k+1})^2]$
θ_ϵ (gain stationnaire)	$ce^{c\frac{t_{k+1}}{\epsilon}(1-\alpha)\log\epsilon}$
$\frac{1}{h}$	$c\epsilon^{4-3\alpha}$
$\frac{\sigma^2 h\epsilon^{2\alpha}}{\epsilon^2+\sigma^2 h^2\epsilon^{2\alpha}}$	$ce^{c\frac{t_{k+1}}{\epsilon}(1-\alpha)\log\epsilon} + c\epsilon^{8-7\alpha}$

D'autre part, on a:

$$E[(X_{k+1} - M_{k+1})^2] \leq ce^{-c\frac{t_{k+1}}{\epsilon^{\alpha}}} + c\epsilon^{2-\alpha} .$$

1.3 Construction des Filtres Approchés (*cas h linéaire*)

On supposera encore que $\gamma = 0$ mais on admet maintenant que la fonction b soit non linéaire.

Pour des raisons liées à l'application d'une méthode de changement de probabilités, pour obtenir les estimations de l'erreur quadratique moyenne, on se voit obligé de considerer notre système non linéaire sous la forme suivante:

$$\begin{cases} X_{k+1} = X_k + b(X_k)\Delta t + \sigma\sqrt{\Delta t}\, w_{k+1}, & X_0 = \xi \\ y_{k+1} = hX_k + \frac{\varepsilon}{\sqrt{\Delta t}} v_{k+1}, & y_0 = 0 \end{cases} \tag{32}$$

ξ étant une $v.a.$ de loi de probabilité p_0 telle que:

$$\int |\frac{p_0'}{p_0} p_0(x)| dx < \infty \ , \ p_0 \in C^1 \ .$$

On suppose toujours que Y_0^k est la tribu des observations jusqu'à l'instant t_k :

$$Y_0^k = \sigma(y_0, y_1, \cdots, y_k) \ .$$

On veut étudier la "qualité" de l'approximation

$$M_{k+1} = M_k + b(M_k)\Delta t + \bar{\theta}(y_{k+1} - hM_k) \ , \quad M_0 = m_0 \ , \tag{33}$$

correspondante à une étape de prévision: $\hat{X}_k = E[X_k | Y_0^k]$.

Supposons, en plus, que b est une fonction C^2 à dérivées bornées.

Commençons par estimer $X_k - M_k$.

$$\begin{aligned} X_{k+1} - M_{k+1} &= X_k + b(X_k)\Delta t + \sigma\sqrt{\Delta t} w_{k+1} - M_k \\ &\quad - b(M_k)\Delta t - \bar{\theta}(y_{k+1} - hM_k) \\ &= (X_k - M_k) + [b(X_k) - b(M_k)]\Delta t + \sigma\sqrt{\Delta t} w_{k+1} \\ &\quad - \bar{\theta}(hX_k + \frac{\varepsilon}{\sqrt{\Delta t}} v_{k+1} - hM_k) \\ &= (1 - h\bar{\theta})(X_k - M_k) + [b(X_k) - b(M_k)]\Delta t \\ &\quad + \sigma\sqrt{\Delta t}(w_{k+1} - \bar{\theta}\frac{\varepsilon}{\sigma\Delta t} v_{k+1}) \ . \end{aligned}$$

Puisque

$$b(X_k) = b(M_k) + b'(\xi_k)(X_k - M_k) \ ,$$

cette expression devient:

$$\begin{aligned} X_{k+1} - M_{k+1} &= (1 - h\bar{\theta} + b'(\xi_k)\Delta t)(X_k - M_k) \\ &\quad + \sigma\sqrt{\Delta t}(w_{k+1} - \bar{\theta}\tfrac{\varepsilon}{\sigma\Delta t}v_{k+1}) . \end{aligned} \tag{34}$$

Soit $\Delta t = \varepsilon^\alpha$, $\alpha > 0$.

Tel que dans la section précédente (cas linéaire) on va considerer 2 situations:

I. On suppose que $\alpha > 1$. Alors, en analogie avec le cas linéaire, on prend $\bar{\theta} = \sigma\dfrac{\Delta t}{\varepsilon}$ dans le schéma (33), i.e.

$$M_{k+1} = M_k + b(M_k)\Delta t + \sigma\frac{\Delta t}{\varepsilon}(y_{k+1} - hM_k) , \quad M_0 = m_0 . \tag{35}$$

I.1. Estimation de $X_k - M_k$.

L'expréssion (34) devient alors:

$$\begin{aligned} X_{k+1} - M_{k+1} &= (1 - \sigma h\frac{\Delta t}{\varepsilon} + b'(\xi_k)\Delta t)(X_k - M_k) \\ &\quad + \sigma\sqrt{\Delta t}(w_{k+1} - v_{k+1}) . \end{aligned} \tag{36}$$

Supposons que b' est bornée, $|b'| \leq c_b \ \forall x$. Vu que $\alpha > 1$ et $X_k - M_k$ et $w_{k+1} - v_{k+1}$ sont indépendants,

$$E[(X_{k+1} - M_{k+1})^2] \leq (1 - \sigma h\frac{\Delta t}{\varepsilon} + c_b\Delta t)^2 E[(X_k - M_k)^2] + 2\sigma^2\Delta t .$$

Soit

$$1 - A \triangleq (1 - \sigma h\tfrac{\Delta t}{\varepsilon} + c_b\Delta t)^2$$

$$B \triangleq 2\sigma^2\Delta t .$$

Avec ces notations,

$$\begin{aligned} E[(X_{k+1} - M_{k+1})^2] &\leq (1 - A)^{k+1}E[(X_0 - M_0)^2] + B\sum_{i=0}^{k}(1 - A)^i \\ &\leq (1 - A)^{k+1}E[(X_0 - M_0)^2] + \frac{B}{A} . \end{aligned} \tag{37}$$

Or,

$$\begin{aligned} A &= 1-(1-\sigma h\frac{\Delta t}{\varepsilon}+c_b\Delta t)^2 \\ &= (\sigma h\frac{\Delta t}{\varepsilon}-c_b\Delta t)(2-\sigma h\frac{\Delta t}{\varepsilon}+c_b\Delta t) \\ &= O(\frac{\Delta t}{\varepsilon}) \text{ et } A \geq c\frac{\Delta t}{\varepsilon} \end{aligned}$$

donc

$$\frac{B}{A} = O(\varepsilon)$$

et (37) devient:

$$\boxed{E[(X_{k+1}-M_{k+1})^2] \leq c\exp\{-c\tfrac{t_{k+1}}{\varepsilon}\}+c\varepsilon} \tag{38}$$

ou, plus précisement,

$$E[(X_k-M_k)^2] \leq c(1-A)^k + c\varepsilon \tag{39}$$

I.2. Estimation de $\hat{X}_k - M_k$.

Théorème 1.4 *Le schéma (35) vérifie:*

$$\hat{X}_k - M_k = \tilde{O}(\varepsilon^{\alpha-\frac{1}{2}} \vee \varepsilon^{\frac{3}{2}})$$

au sens où

$$E[|\hat{X}_k - M_k|] \leq c(1\vee\varepsilon^{2-\alpha})\exp\{-c\frac{t_k}{\varepsilon}\} + c(\varepsilon^{\alpha-\frac{1}{2}} \vee \varepsilon^{\frac{3}{2}})\,. \quad [1]$$

---

[1] On utilisera la notation:

$$\Psi_k = \tilde{O}\varepsilon^q\,,$$

où $\{\Psi\}_k$ est un processus dépéndant de ε et $q \geq 0$, pour signifier que:

$$E[|\Psi_k|] \leq c_0(1\vee\varepsilon^{c(q)})\exp\{-c_1\frac{t_k}{\varepsilon}\} + c_2\varepsilon^q\ ;\ c_0, c_1, c_2 > 0\,.$$

On utilisera la notation:

$$\Psi_k = O(\varepsilon^q)$$

pour signifier que:

$$E[|\Psi_k|] \leq c_0\exp\{-c_1\frac{t_k}{\varepsilon}\} + c_2\varepsilon^q.$$

Preuve

Elle sera divisée en plusieures parties, utilisant des changement de probabilités, une vèrsion discrète du Theoréme de Girsanov et la dérivation par rapport à la condition initiale.

Changement de probabilités.

(a) Le 1^{ier} changement de probabilités affectera la loi de v.

On considère la probabilité $\dot{P}$ définie par:

$$\frac{d\dot{P}}{dP}\Big|_{\mathcal{F}_k} = L_k^{-1} ,$$

avec

$$L_k^{-1} = \exp\{\sum_{i=1}^{k}(-\frac{\sqrt{\Delta t}}{\varepsilon}hX_{i-1})v_i - \frac{1}{2}\sum_{i=1}^{k}(-\frac{\sqrt{\Delta t}}{\varepsilon}hX_{i-1})^2\} .$$

Soit

$$\frac{\sqrt{\Delta t}}{\varepsilon}y_k = v_k - (-\frac{\sqrt{\Delta t}}{\varepsilon}hX_{k-1}) ,$$

où X_{k-1} est $\mathcal{F}_{k-1}$- mesurable.

Ça donne:

$$\begin{aligned} L_k &= \exp\{\sum_{i=1}^{k}(\frac{\sqrt{\Delta t}}{\varepsilon}hX_{i-1})(\frac{\sqrt{\Delta t}}{\varepsilon}y_i - \frac{\sqrt{\Delta t}}{\varepsilon}hX_{i-1}) \\ &\quad +\frac{1}{2}\sum_{i=1}^{k}(-\frac{\sqrt{\Delta t}}{\varepsilon}hX_{i-1})^2\} \\ &= \exp\{\frac{\Delta t}{\varepsilon^2}(h\sum_{i=1}^{k}X_{i-1}y_i - \sum_{i=1}^{k}h^2X_{i-1}^2) + \frac{h^2\Delta t}{2\varepsilon^2}\sum_{i=1}^{k}X_{i-1})^2\} \\ &= \exp\{h\frac{\Delta t}{\varepsilon^2}(\sum_{i=1}^{k}X_{i-1}y_i - \frac{h}{2}\sum_{i=1}^{k}X_{i-1}^2)\} . \end{aligned}$$

D'après la vérsion discrète du Théorème de Girsanov (voir l'annexe A),

> sous $\dot{P}$ (probabilité de référance), w_k et $\frac{\sqrt{\Delta t}}{\varepsilon}y_k$ sont des $\mathcal{F}_k$-b.b. gaussiens indépendants. ($\dot{P}$ est équivalent à P dans chaque $\mathcal{F}_k$.)

(b) Le $2^{ième}$ changement de probabilités va affecter la loi de w.

On définit la nouvelle probabilité $\tilde{P}$ par:

$$\frac{d\tilde{P}}{d\dot{P}}\Big|_{\mathcal{F}_k} = \Lambda_k^{-1} ,$$

étant

$$\Lambda_k^{-1} = \exp\{\sum_{i=1}^{k} h\frac{\sqrt{\Delta t}}{\varepsilon}(X_{i-1} - M_{i-1})w_i - \frac{1}{2}\sum_{i=1}^{k}[h\frac{\sqrt{\Delta t}}{\varepsilon}(X_{i-1} - M_{i-1})]^2\} .$$

Soit

$$\tilde{w}_k = w_k - h\frac{\sqrt{\Delta t}}{\varepsilon}(X_{k-1} - M_{k-1}) ,$$

où $X_{k-1} - M_{k-1}$ est $\mathcal{F}_{k-1}$-mesurable.

(On rappelle que:

$$\begin{aligned} X_{k+1} - M_{k+1} &= (1 - \sigma h\frac{\Delta t}{\varepsilon})(X_k - M_k) + [b(X_k) - b(M_k)]\Delta t \\ &\quad +\sigma\sqrt{\Delta t}(w_{k+1} - v_{k+1}) \text{ .)} \end{aligned}$$

Alors

$$\begin{aligned} \Lambda_k &= \exp\{-\sum_{i=1}^{k} h\frac{\sqrt{\Delta t}}{\varepsilon}(X_{i-1} - M_{i-1})[\tilde{w}_i + h\frac{\sqrt{\Delta t}}{\varepsilon}(X_{i-1} - M_{i-1})] \\ &\quad +\frac{1}{2}\sum_{i=1}^{k} h^2\frac{\Delta t}{\varepsilon^2}(X_{i-1} - M_{i-1})^2\} \\ &= \exp\{-h\frac{\sqrt{\Delta t}}{\varepsilon}\sum_{i=1}^{k}(X_{i-1} - M_{i-1})\tilde{w}_i - h^2\frac{\Delta t}{\varepsilon^2}(X_{i-1} - M_{i-1})^2 \\ &\quad +\frac{h^2\Delta t}{2\varepsilon^2}\sum_{i=1}^{k}(X_{i-1} - M_{i-1})^2\} \\ &= \exp\{-h\frac{\sqrt{\Delta t}}{\varepsilon}\sum_{i=1}^{k}(X_{i-1} - M_{i-1})\tilde{w}_i - \frac{h^2\Delta t}{2\varepsilon^2}\sum_{i=1}^{k}(X_{i-1} - M_{i-1})^2\} \end{aligned}$$

D'après la version discrète du Théorème de Girsanov,

Sous $\tilde{P}$, $\tilde{w}_k$ et $\frac{\sqrt{\Delta t}}{\varepsilon}y_k$ sont des b.b. gauss. ind. et X_k vérifie:

$$X_{k+1} = \sigma h\frac{\Delta t}{\varepsilon}(X_k - M_k) + X_k + b(X_k)\Delta t + \sigma\sqrt{\Delta t}\tilde{w}_{k+1} . \quad (40)$$

La densité de P par rapport à $\tilde{P}$ est $L_k\Lambda_k$ donc:

$\forall$ *v.a.* ψ P-intégrable et $\mathcal{F}_k$-mesurable,

$$E[\psi|Y_k] = \frac{\tilde{E}[\psi L_k\Lambda_k|Y_k]}{\tilde{E}[L_k\Lambda_k|Y_k]} .$$

Or,

$$\begin{aligned} L_k\Lambda_k &= \exp\{h\frac{\Delta t}{\varepsilon^2}(\sum_{i=1}^k X_{i-1}y_i - \frac{h}{2}\sum_{i=1}^k X_{i-1}^2)\} \\ &\quad . \exp\{-h\frac{\sqrt{\Delta t}}{\varepsilon}\sum_{i=1}^k (X_{i-1}-M_{i-1})\tilde{w}_i - \frac{h^2}{2}\frac{\Delta t}{\varepsilon^2}\sum_{i=1}^k (X_{i-1}-M_{i-1})^2\} . \end{aligned}$$

Soit

$$\begin{aligned} S &\triangleq h\frac{\Delta t}{\varepsilon^2}\sum_{i=1}^k X_{i-1}y_i - \frac{h^2}{2}\frac{\Delta t}{\varepsilon^2}\sum_{i=1}^k X_{i-1}^2 - h\frac{\sqrt{\Delta t}}{\varepsilon}\sum_{i=1}^k (X_{i-1}-M_{i-1})\tilde{w}_i \\ &\quad -\frac{h^2}{2}\frac{\Delta t}{\varepsilon^2}\sum_{i=1}^k (X_{i-1}-M_{i-1})^2 \end{aligned}$$

i.e.

$$L_k\Lambda_k = \exp S .$$

Mais,

$$\begin{aligned} X_i - M_i &= (X_{i-1}-M_{i-1}) + [b(X_{i-1}) - b(M_{i-1})]\Delta t \\ &\quad +\sigma\sqrt{\Delta t}w_i - \sigma\frac{\Delta t}{\varepsilon}(y_i - hM_{i-1}) \\ &= (X_{i-1}-M_{i-1} + [b(X_{i-1}) - b(M_{i-1})]\Delta t \\ &\quad +\sigma\sqrt{\Delta t}\tilde{w}_i + \sigma h\frac{\Delta t}{\varepsilon}(X_{i-1}-M_{i-1}) - \sigma\frac{\Delta t}{\varepsilon}(y_i - hM_{i-1}) \\ &= (1+\sigma h\frac{\Delta t}{\varepsilon})(X_{i-1}-M_{i-1}) + [b(X_{i-1}) - b(M_{i-1})]\Delta t \\ &\quad +\sigma\sqrt{\Delta t}\tilde{w}_i - \sigma\frac{\Delta t}{\varepsilon}(y_i - hM_{i-1}) \end{aligned}$$

d'où

$$\begin{aligned} \tilde{w}_i &= \frac{1}{\sigma\sqrt{\Delta t}}(X_i - M_i) - \frac{1}{\sigma\sqrt{\Delta t}}(1+\sigma h\frac{\Delta t}{\varepsilon})(X_{i-1}-M_{i-1}) \\ &\quad -\frac{\sqrt{\Delta t}}{\sigma}[b(X_{i-1}) - b(M_{i-1})] + \frac{\sqrt{\Delta t}}{\varepsilon}(y_i - hM_{i-1}) . \end{aligned}$$

Donc

$$
\begin{aligned}
S \;=\;& \frac{h\Delta t}{\varepsilon^2}\sum_{i=1}^{k} X_{i-1}y_i - \frac{h^2\Delta t}{2\varepsilon^2}\sum_{i=1}^{k} X_{i-1}^2 - \frac{h}{\sigma\varepsilon}\sum_{i=1}^{k}(X_{i-1}-M_{i-1})(X_i-M_i) \qquad (41)\\
&+\frac{h}{\sigma\varepsilon}(1+\sigma h\frac{\Delta t}{\varepsilon})\sum_{i=1}^{k}(X_{i-1}-M_{i-1})^2\\
&+\frac{h\Delta t}{\sigma\varepsilon}\sum_{i=1}^{k}(X_{i-1}-M_{i-1})[b(X_{i-1})-b(M_{i-1})]\\
&-\frac{h\Delta t}{\varepsilon^2}\sum_{i=1}^{k}(X_{i-1}-M_{i-1})(y_i-hM_{i-1}) - \frac{h^2\Delta t}{2\varepsilon^2}\sum_{i=1}^{k}(X_{i-1}-M_{i-1})^2\\
=\;& -\frac{h^2\Delta t}{2\varepsilon^2}\sum_{i=1}^{k} X_{i-1}^2 - \frac{h}{\sigma\varepsilon}\sum_{i=1}^{k}(X_{i-1}-M_{i-1})(X_i-M_i)\\
&+\frac{h}{\sigma\varepsilon}(1+\sigma h\frac{\Delta t}{\varepsilon})\sum_{i=1}^{k}(X_{i-1}-M_{i-1})^2\\
&+\frac{h\Delta t}{\sigma\varepsilon}\sum_{i=1}^{k}(X_{i-1}-M_{i-1})[b(X_{i-1})-b(M_{i-1})]\\
&+\frac{h^2\Delta t}{\varepsilon^2}\sum_{i=1}^{k} X_{i-1}M_{i-1} + \frac{h\Delta t}{\varepsilon^2}\sum_{i=1}^{k} M_{i-1}(y_i-hM_{i-1}) - \frac{h^2\Delta t}{2\varepsilon^2}\sum_{i=1}^{k}(X_{i-1}-M_{i-1})^2\\
=\;& \frac{h\Delta t}{\sigma\varepsilon}\sum_{i=1}^{k}(X_{i-1}-M_{i-1})[b(X_{i-1})-b(M_{i-1})] - \frac{h}{\sigma\varepsilon}\sum_{i=1}^{k}(X_{i-1}-M_{i-1})(X_i-M_i)\\
&+\frac{h}{\sigma\varepsilon}\sum_{i=1}^{k}(X_{i-1}-M_{i-1})^2 + \frac{h^2\Delta t}{2\varepsilon^2}\sum_{i=1}^{k}[-X_{i-1}^2+(X_{i-1}-M_{i-1})^2+2X_{i-1}M_{i-1}]\\
&+\frac{h\Delta t}{\varepsilon^2}\sum_{i=1}^{k} M_{i-1}(y_i-hM_{i-1})\\
=\;& \frac{h\Delta t}{\sigma\varepsilon}\sum_{i=1}^{k}(X_{i-1}-M_{i-1})[b(X_{i-1})-b(M_{i-1})] - \frac{h}{\sigma\varepsilon}\sum_{i=1}^{k}(X_{i-1}-M_{i-1})(X_i-M_i)\\
&+\frac{h}{\sigma\varepsilon}\sum_{i=1}^{k}(X_{i-1}-M_{i-1})^2 + \frac{h^2\Delta t}{2\varepsilon^2}\sum_{i=1}^{k} M_{i-1}^2 + \frac{h\Delta t}{\varepsilon^2}\sum_{i=1}^{k} M_{i-1}(y_i-hM_{i-1})\\
=\;& -\frac{h}{\sigma\varepsilon}\sum_{i=1}^{k}(X_{i-1}-M_{i-1})[(X_i-M_i)-(X_{i-1}-M_{i-1})]\\
&+\frac{h\Delta t}{\sigma\varepsilon}\sum_{i=1}^{k}(X_{i-1}-M_{i-1})[b(X_{i-1})-b(M_{i-1})]\\
&+\frac{h\Delta t}{\varepsilon^2}\sum_{i=1}^{k} M_{i-1}y_i - \frac{h^2\Delta t}{2\varepsilon^2}\sum_{i=1}^{k} M_{i-1}^2\,.
\end{aligned}
$$

Soit $u_i = X_i - M_i$. On a:

$$\begin{aligned}\sum_i (u_i^2 - u_{i-1}^2) &= \sum_i (u_i + u_{i-1})(u_i - u_{i-1}) \\ &= 2\sum_i u_{i-1}(u_i - u_{i-1}) + \sum_i (u_i - u_{i-1})^2\end{aligned}$$

i.e.

$$\sum_i u_{i-1}(u_i - u_{i-1}) = \frac{1}{2}\sum_i (u_i^2 - u_{i-1}^2) - \frac{1}{2}\sum_i (u_i - u_{i-1})^2 .$$

Alors

$$\begin{aligned}S &= -\frac{h}{2\sigma\varepsilon}\sum_{i=1}^{k}[(X_i - M_i)^2 - (X_{i-1} - M_{i-1})^2] & (42)\\ &+\frac{h\Delta t}{\sigma\varepsilon}\sum_{i=1}^{k}(X_{i-1} - M_{i-1})[b(X_{i-1}) - b(M_{i-1})] \\ &+\frac{h\Delta t}{\varepsilon^2}\sum_{i=1}^{k} M_{i-1}y_i - \frac{h^2\Delta t}{2\varepsilon^2}\sum_{i=1}^{k} M_{i-1}^2 \\ &+\frac{h}{2\sigma\varepsilon}\sum_{i=1}^{k}[(X_i - M_i) - (X_{i-1} - M_{i-1})]^2 \\ & & (43)\end{aligned}$$

Remarquons que les $3^{ième}$ et $4^{ième}$ tèrmes du second membre de cette expression sont Y_0^k-adaptés et disparaîtront donc dans la normalisation.

Dérivation par rapport à la condition initiale .

On considére les variables qui interviennent dans nos calculs comme des fonctions de X_0 (condition initiale) et des processus $\tilde{w}_k$ et y_k.

Soit $\psi_k = \psi(X_0, \tilde{w}_k, y_k)$.

On veut dériver (40) et (43).

Définissons alors les processus

$$\begin{aligned}Z_k &\triangleq \frac{\partial X_k}{\partial X_0}\ , \ Z_0 \equiv 1 \\ Z_{nk} &\triangleq \frac{Z_k}{Z_n}\ , \ Z_{0k} = Z_k\end{aligned}$$

et <u>fixons k</u>.

Pour $n \geq k$, Z_{nk} vérifie l'expression:

$$\begin{aligned} Z_{nk} &= \prod_{j=k}^{n-1}(1+\sigma h\frac{\Delta t}{\varepsilon}+\Delta t b'(X_j))^{-1} \\ Z_{kk} &= 1\,. \end{aligned} \tag{44}$$

Preuve :
De (40) vient que:

$$X_{i+1} = \sigma h\frac{\Delta t}{\varepsilon}(X_i - M_i) + X_i + b(X_i)\Delta t + \sigma\sqrt{\Delta t}\tilde{w}_{i+1}$$

donc

$$\frac{\partial X_{i+1}}{\partial X_0} = \sigma h\frac{\Delta t}{\varepsilon}\frac{\partial X_i}{\partial X_0} + \frac{\partial X_i}{\partial X_0} + \Delta t b'(X_i)\frac{\partial X_i}{\partial X_0}$$

i.e.

$$\begin{aligned} Z_{i+1} &= (1+\sigma h\frac{\Delta t}{\varepsilon}+\Delta t b'(X_i))Z_i \\ Z_{i,i+1} &= \frac{Z_{i+1}}{Z_i} = 1+\sigma h\frac{\Delta t}{\varepsilon}+\Delta t b'(X_i)\,. \end{aligned}$$

Par récurrence,

$$Z_{i,i+m} = \prod_{j=i}^{i+m-1}(1+\sigma h\frac{\Delta t}{\varepsilon}+\Delta t b'(X_j))\,.$$

Pour $n \geq k$,

$$Z_{nk} = \prod_{j=k}^{n-1}(1+\sigma h\frac{\Delta t}{\varepsilon}+\Delta t b'(X_j))^{-1}\,.$$

■

On a la majoration suivante:

Si b' est bornée,

$$Z_{nk} \leq \prod_{j=k}^{n-1}(1+\sigma h\frac{\Delta t}{\varepsilon}-c_b\Delta t)^{-1} = (1+\sigma h\frac{\Delta t}{\varepsilon}-c_b\Delta t)^{-(n-k)} \tag{45}$$

i.e.

$$Z_{nk} \leq c\exp\{-c(n-k)\frac{\frac{\Delta t}{\varepsilon}}{1+c\frac{\Delta t}{\varepsilon}}\} \leq c\exp\{-c\frac{t_{n-k}}{\varepsilon}\}\,.$$

D'autre part,

$$
\begin{aligned}
\frac{\partial}{\partial X_0}\log(L_k\Lambda_k) &= -\frac{h}{\sigma\epsilon}\sum_{i=1}^{k}[(X_i - M_i)\frac{\partial X_i}{\partial X_0} - (X_{i-1} - M_{i-1})\frac{\partial X_{i-1}}{\partial X_0}] \\
&\quad + \frac{h\Delta t}{\sigma\epsilon}\sum_{i=1}^{k}[\frac{\partial X_{i-1}}{\partial X_0}[b(X_{i-1}) - b(M_{i-1})] \\
&\quad + (X_{i-1} - M_{i-1})b'(X_{i-1})\frac{\partial X_{i-1}}{\partial X_0}] \\
&\quad + \frac{h}{\sigma\epsilon}\sum_{i=1}^{k}[(X_i - M_i) - (X_{i-1} - M_{i-1})](\frac{\partial X_i}{\partial X_0} - \frac{\partial X_{i-1}}{\partial X_0}) \\
&= \frac{h}{\sigma\epsilon}\sum_{i=1}^{k}(X_i - M_i)(-\frac{\partial X_i}{\partial X_0} + \frac{\partial X_i}{\partial X_0} - \frac{\partial X_{i-1}}{\partial X_0}) \\
&\quad + \frac{h}{\sigma\epsilon}\sum_{i=1}^{k}(X_{i-1} - M_{i-1})(\frac{\partial X_{i-1}}{\partial X_0} + \Delta t b'(X_{i-1})\frac{\partial X_{i-1}}{\partial X_0} \\
&\qquad\qquad - \frac{\partial X_i}{\partial X_0} + \frac{\partial X_{i-1}}{\partial X_0}) \\
&\quad + \frac{h\Delta t}{\sigma\epsilon}\sum_{i=1}^{k}\frac{\partial X_{i-1}}{\partial X_0}[b(X_{i-1}) - b(M_{i-1})] \\
&= -\frac{h}{\sigma\epsilon}\sum_{i=1}^{k}(X_i - M_i)Z_{i-1} \\
&\quad - \frac{h}{\sigma\epsilon}\sum_{i=1}^{k}(X_{i-1} - M_{i-1})[Z_i - 2Z_{i-1} - \Delta t b'(X_{i-1})Z_{i-1}] \\
&\quad + \frac{h\Delta t}{\sigma\epsilon}\sum_{i=1}^{k}[b(X_{i-1}) - b(M_{i-1})]Z_{i-1}\,.
\end{aligned}
$$

Expression asymptotique pour $X_k - M_k$.

L'égalité précédente nous donne:

$$
\begin{aligned}
\frac{h}{\sigma\epsilon}(X_k - M_k)Z_{k-1} &= -\frac{h}{\sigma\epsilon}\sum_{i=1}^{k-1}(X_i - M_i)Z_{i-1} - \frac{\partial}{\partial X_0}\log(L_k\Lambda_k) \\
&\quad - \frac{h}{\sigma\epsilon}\sum_{i=1}^{k}(X_{i-1} - M_{i-1})(Z_i - 2Z_{i-1} - \Delta t b'(X_{i-1})Z_{i-1}) \\
&\quad + \frac{h\Delta t}{\sigma\epsilon}\sum_{i=1}^{k}[b(X_{i-1}) - b(M_{i-1})]Z_{i-1}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{h}{\sigma\varepsilon}\sum_{i=2}^{k}(X_{i-1}-M_{i-1})Z_{i-2} - \frac{\partial}{\partial X_0}\log(L_k\Lambda_k) \\
&\quad -\frac{h}{\sigma\varepsilon}\sum_{i=1}^{k}(X_{i-1}-M_{i-1})(Z_i - 2Z_{i-1} - \Delta t b'(X_{i-1})Z_{i-1}) \\
&\quad +\frac{h\Delta t}{\sigma\varepsilon}\sum_{i=1}^{k}[b(X_{i-1}) - b(M_{i-1})]Z_{i-1} \\
&= -\frac{h}{\sigma\varepsilon}\sum_{i=1}^{k}(X_{i-1}-M_{i-1})(Z_i - 2Z_{i-1} - \Delta t b'(X_{i-1})Z_{i-1} + Z_{i-2}) \\
&\quad +\frac{h\Delta t}{\sigma\varepsilon}\sum_{i=1}^{k}[b(X_{i-1}) - b(M_{i-1})]Z_{i-1} - \frac{\partial}{\partial X_0}\log(L_k\Lambda_k)\ , \\
&\quad \text{avec } Z_{-1} \triangleq 0\ .
\end{aligned}
$$

Donc

$$
\begin{aligned}
X_k - M_k &= -\sum_{i=1}^{k}(X_{i-1}-M_{i-1})[\frac{Z_i}{Z_{k-1}} - 2\frac{Z_{i-1}}{Z_{k-1}} - \Delta t b'(X_{i-1})\frac{Z_{i-1}}{Z_{k-1}} + \frac{Z_{i-2}}{Z_{k-1}}] \\
&\quad +\Delta t\sum_{i=1}^{k}[b(X_{i-1}) - b(M_{i-1})]\frac{Z_{i-1}}{Z_{k-1}} - \frac{\sigma\varepsilon}{h}\frac{\partial}{\partial X_0}\log(L_k\Lambda_k)\frac{1}{Z_{k-1}} \\
&= -\sum_{i=1}^{k}(X_{i-1}-M_{i-1})[Z_{k-1,i} - 2Z_{k-1,i-1} - \Delta t b'(X_{i-1})Z_{k-1,i-1} + Z_{k-1,i-2}] \\
&\quad +\Delta t\sum_{i=1}^{k}[b(X_{i-1}) - b(M_{i-1})]Z_{k-1,i-1} - \frac{\sigma\varepsilon}{h}\frac{\partial}{\partial X_0}\log(L_k\Lambda_k)Z_{k-1,0} \\
&= \sum_{i=1}^{k}\{(X_{i-1}-M_{i-1})[(Z_{k-1,i-1} - Z_{k-1,i-2}) - (Z_{k-1,i} - Z_{k-1,i-1})] \\
&\qquad +\Delta t[b(X_{i-1}) - b(M_{i-1}) + b'(X_{i-1})(X_{i-1}-M_{i-1})]Z_{k-1,i-1}\} \\
&\quad -\frac{\sigma\varepsilon}{h}\frac{\partial}{\partial X_0}\log(L_k\Lambda_k)Z_{k-1,0}\ . \qquad (46)
\end{aligned}
$$

Or,

$$
\begin{aligned}
Z_{k-1,i} - Z_{k-1,i-1} &= (Z_{i-1,i} - 1)Z_{k-1,i-1} \\
&= (\sigma h\frac{\Delta t}{\varepsilon} + \Delta t b'(X_{i-1}))Z_{k-1,i-1} \qquad (47) \\
Z_{k-1,i-1} - Z_{k-1,i-2} &= (1 - Z_{i-1,i-2})Z_{k-1,i-1} \\
&= \left[1 - \frac{1}{1 + \sigma h\frac{\Delta t}{\varepsilon} + \Delta t b'(X_{i-2})}\right] Z_{k-1,i-1}
\end{aligned}
$$

$$= \frac{\sigma h\frac{\Delta t}{\varepsilon} + \Delta t b'(X_{i-2})}{1 + \sigma h\frac{\Delta t}{\varepsilon} + \Delta t b'(X_{i-2})} Z_{k-1,i-1}, \text{ si } i \geq 2 \quad (48)$$

donc

• Pour $i \geq 2$,

$$\begin{aligned}
(Z_{k-1,i-1} - Z_{k-1,i-2}) - (Z_{k-1,i} - Z_{k-1,i-1}) &= \frac{\sigma h\frac{\Delta t}{\varepsilon} + \Delta t b'(X_{i-2})}{1 + \sigma h\frac{\Delta t}{\varepsilon} + \Delta t b'(X_{i-2})} Z_{k-1,i-1} \\
&\quad - (\sigma h\frac{\Delta t}{\varepsilon} + \Delta t b'(X_{i-1})) Z_{k-1,i-1} \\
&= \frac{Num}{1 + \sigma h\frac{\Delta t}{\varepsilon} + \Delta t b'(X_{i-2})} Z_{k-1,i-1} ,
\end{aligned}$$

où

$$\begin{aligned}
Num &\triangleq \sigma h\frac{\Delta t}{\varepsilon} + \Delta t b'(X_{i-2}) - (\sigma h\frac{\Delta t}{\varepsilon} + \Delta t b'(X_{i-1}))(1 + \sigma h\frac{\Delta t}{\varepsilon} + \Delta t b'(X_{i-2})) \\
&= \sigma h\frac{\Delta t}{\varepsilon} + \Delta t b'(X_{i-2}) - \sigma h\frac{\Delta t}{\varepsilon} - \sigma^2 h^2\frac{\Delta t^2}{\varepsilon^2} - \sigma h\frac{\Delta t^2}{\varepsilon} b'(X_{i-2}) \\
&\quad - \Delta t b'(X_{i-1}) - \sigma h\frac{\Delta t^2}{\varepsilon} b'(X_{i-1}) - \Delta t^2 b'(X_{i-1}) b'(X_{i-2}) \\
&= -\sigma^2 h^2\frac{\Delta t^2}{\varepsilon^2} - \Delta t[b'(X_{i-1}) - b'(X_{i-2})] - \sigma h\frac{\Delta t^2}{\varepsilon}[b'(X_{i-1}) + b'(X_{i-2})] \\
&\quad - \Delta t^2 b'(X_{i-1}) b'(X_{i-2}) \\
&= O(\Delta t^{\frac{3}{2}} \vee \frac{\Delta t^2}{\varepsilon^2}) \text{ , dans } L^r \text{ et } Num = O(\Delta t \vee \frac{\Delta t^2}{\varepsilon^2}) \text{ , dans } L^\infty .
\end{aligned}$$

• Pour $i = 1$,

$$\begin{aligned}
(Z_{k-1,i-1} - Z_{k-1,i-2}) - (Z_{k-1,i} - Z_{k-1,i-1}) &= Z_{k-1,0} - (Z_{k-1,1} - Z_{k-1,0}) \\
&= Z_{k-1,0} - (\sigma h\frac{\Delta t}{\varepsilon} + \Delta t b'(X_0)) Z_{k-1,0} \\
&= (1 - \sigma h\frac{\Delta t}{\varepsilon} - \Delta t b'(X_0)) Z_{k-1,0} .
\end{aligned}$$

L'égalité (46) devient:

$$X_k - M_k = \sum_{i=2}^{k} (X_{i-1} - M_{i-1})[\frac{Num}{1 + \sigma h\frac{\Delta t}{\varepsilon} + \Delta t b'(X_{i-2})} + \Delta t(b'(\xi_{i-1}) + b'(X_{i-1}))] Z_{k-1,i-1}$$

$$+(X_0 - M_0)(1 - \sigma h\frac{\delta t}{\varepsilon} - \Delta t b'(X_0))Z_{k-1,0} - \frac{\sigma\varepsilon}{h}\frac{\partial}{\partial X_0}\log(L_k\Lambda_k)Z_{k-1,0}\ . \quad (49)$$

Soit

$$\begin{aligned}
\phi_i &= \frac{Num}{1 + \sigma h\frac{\Delta t}{\varepsilon} + \Delta t b'(X_{i-2})} + \Delta t(b'(\xi_{i-1}) + b'(X_{i-1}))\ ,\ \text{pour } i \geq 2 \\
\phi_1 &= 1 - \sigma h\frac{\Delta t}{\varepsilon} - \Delta t b'(X_0)\ .
\end{aligned}$$

Alors,

$$|\phi_i| \leq \rho_{i-1}\ ,\ i \geq 2$$

où

$$\begin{aligned}
\rho_{i-1} &= \frac{|Num|}{1 + \sigma h\frac{\Delta t}{\varepsilon} + \Delta t b'(X_{i-2})} + 2c_b\Delta t \\
&= O(\frac{\Delta t^2}{\varepsilon^2} \vee \Delta t)\ ,\ \text{dans } L^r \text{ et dans } L^\infty\ .
\end{aligned}$$

Soit ρ *t.q.* : $\rho_i \leq \rho\ \forall i \geq 1$.
D'après (45),

$$Z_{nk} \leq (1 + \sigma h\frac{\Delta t}{\varepsilon} - c_b\Delta t)^{-(n-k)}\ ,$$

donc

$$\begin{aligned}
\sum_{i=1}^{k} E[|X_{i-1} - M_{i-1}||\phi_i|Z_{k-1,i-1}] \leq\ & \rho\sum_{i=2}^{k}(1 + \sigma h\frac{\Delta t}{\varepsilon} - c_b\Delta t)^{-(k-i)}E[|X_{i-1} - M_{i-1}|] \\
& +(1 - \sigma h\frac{\delta t}{\varepsilon} + c_b\Delta t)(1 + \sigma h\frac{\delta t}{\varepsilon} - c_b\Delta t)^{-(k-1)} \\
& .E[|X_0 - M_0|]\ .
\end{aligned}$$

Puisque

$$\begin{aligned}
E[|X_i - M_i|] &\leq \sqrt{E[(X_i - M_i)^2]} \\
&\leq [c(1 - A)^i + c\varepsilon]^{\frac{1}{2}}\ \text{(voir le résultat (39))}
\end{aligned}$$

on a:

$$\begin{aligned}
\sum_{i=1}^{k} E[|X_{i-1} - M_{i-1}||\phi_i|Z_{k-1,i-1}] \leq\ & c\rho\sum_{i=2}^{k}(1 + \sigma h\frac{\Delta t}{\varepsilon} - c_b\Delta t)^{-(k-i)} \\
& \cdot [(1 - A)^{i-1} + c\varepsilon]^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
& +(1-c\frac{\Delta t}{\varepsilon})^k E[|X_0 - M_0|] \\
\leq\ & c\rho \sum_{i=2}^{k} (1+\sigma h \frac{\Delta t}{\varepsilon} - c_b \Delta t)^{-(k-i)} \\
& \qquad \cdot [((1-A)^{\frac{1}{2}})^{i-1} + c\sqrt{\varepsilon}] \\
& +(1-c\frac{\Delta t}{\varepsilon})^k E[|X_0 - M_0|] \\
=\ & c\rho \{ \sum_{i=2}^{k} (1+\sigma h \frac{\Delta t}{\varepsilon} - c_b \Delta t)^{-(k-i)} \\
& \qquad \cdot (1-\sigma h \frac{\Delta t}{\varepsilon} + c_b \Delta t)^{i-1} \\
& \quad + c\sqrt{\varepsilon} \sum_{i=2}^{k} (1+\sigma h \frac{\Delta t}{\varepsilon} - c_b \Delta t)^{-(k-i)} \} \\
& +(1-c\frac{\Delta t}{\varepsilon})^k E[|X_0 - M_0|] \\
=\ & c\rho \{ (1+\sigma h \frac{\Delta t}{\varepsilon} - c_b \Delta t)^{-k+1} \\
& \quad \cdot \sum_{i=1}^{k-1} [(1+\sigma h \frac{\Delta t}{\varepsilon} - c_b \Delta t)(1-\sigma h \frac{\Delta t}{\varepsilon} + c_b \Delta t)]^i \\
& \quad + c\sqrt{\varepsilon} \sum_{i=0}^{k-2} (1+\sigma h \frac{\Delta t}{\varepsilon} - c_b \Delta t)^{-i} \} \\
& +(1-c\frac{\Delta t}{\varepsilon})^k E[|X_0 - M_0|] \\
\leq\ & c\rho \{ (1+\sigma h \frac{\Delta t}{\varepsilon} - c_b \Delta t)^{-k+1} \sum_{i=1}^{k-1} [1 - (\sigma h \frac{\Delta t}{\varepsilon} - c_b \Delta t)^2]^i \\
& \quad + c\sqrt{\varepsilon} \frac{1}{1-(1+\sigma h \frac{\Delta t}{\varepsilon} - c_b \Delta t)^{-1}} \} \\
& +(1-c\frac{\Delta t}{\varepsilon})^k E[|X_0 - M_0|] \\
\leq\ & c\rho \{ (1+\sigma h \frac{\Delta t}{\varepsilon} - c_b \Delta t)^{-k+1} \frac{1-(\sigma h \frac{\Delta t}{\varepsilon} - c_b \Delta t)^2}{(\sigma h \frac{\Delta t}{\varepsilon} - c_b \Delta t)^2} \\
& \quad + c\sqrt{\varepsilon} \cdot \frac{1+\sigma h \frac{\Delta t}{\varepsilon} - c_b \Delta t}{\sigma h \frac{\Delta t}{\varepsilon} - c_b \Delta t} \}
\end{aligned}
$$

$$+(1-c\frac{\Delta t}{\varepsilon})^k E[|X_0 - M_0|]$$

$$\leq \quad c(1+c\rho\frac{\varepsilon^2}{\Delta t^2})\exp\{-c\frac{t_{k-1}}{\varepsilon}\} + c\rho\frac{\varepsilon^{\frac{3}{2}}}{\Delta t}\,. \tag{50}$$

On aura besoin, une fois encore, de considerer deux cas:

- Si $\alpha \geq 2$, on a $\rho = c\varepsilon^\alpha$ et (50) devient alors:

$$\sum_{i=1}^{k} E[|X_{i-1} - M_{i-1}||\phi_i|Z_{k-1,i-1}] \leq c\varepsilon^{2-\alpha}\exp\{-c\frac{t_{k-1}}{\varepsilon}\} + c\varepsilon^{\frac{3}{2}}$$

donc

$$X_k - M_k + \frac{\sigma\varepsilon}{h}\frac{\partial}{\partial X_0}\log(L_k\Lambda_k)Z_{k-1,0} = \tilde{O}(\varepsilon^{\frac{3}{2}})\,.$$

Prenons l'espérance conditionnelle par rapport à Y_0^k.
Si on montre que:

$$E[\frac{\partial}{\partial X_0}\log(L_k\Lambda_k)Z_{k-1,0}|Y_0^k] = O(\sqrt{\varepsilon}) \tag{51}$$

alors

$$\hat{X}_k - M_k = \tilde{O}(\varepsilon^{\frac{3}{2}})\,,$$

au sens où

$$E[|\hat{X}_k - M_k|] \leq c\varepsilon^{2-\alpha}\exp\{-\mu\frac{t_k}{\varepsilon}\} + c\varepsilon^{\frac{3}{2}}\ ;\ c,\mu > 0\,.$$

On utilise le lemme 1.6 pour prouver ce résultat.

- Si $\alpha \leq 2$, on a $\rho = c\varepsilon^{2(\alpha-1)}$ et (50) devient alors:

$$\sum_{i=1}^{k} E[|X_{i-1} - M_{i-1}||\phi_i|Z_{k-1,i-1}] \leq c\exp\{-c\frac{t_{k-1}}{\varepsilon}\} + c\varepsilon^{\alpha-\frac{1}{2}}$$

donc

$$X_k - M_k + \frac{\sigma\varepsilon}{h}\frac{\partial}{\partial X_0}\log(L_k\Lambda_k)Z_{k-1,0} = O(\varepsilon^{\alpha-\frac{1}{2}})\,.$$

On utilise le même raisonnement du cas précédent.
Encore une fois, prenons l'espérance conditionnelle par rapport à Y_0^k.

Si on montre que:

$$E[\frac{\partial}{\partial X_0}\log(L_k\Lambda_k)Z_{k-1,0}|Y_0^k] = O(\varepsilon^{\alpha-\frac{3}{2}}) \tag{52}$$

alors

$$\hat{X}_k - M_k = O(\varepsilon^{\alpha-\frac{1}{2}})\ ,$$

au sens où

$$E[|\hat{X}_k - M_k|] \leq c\exp\{-\mu\frac{t_k}{\varepsilon}\} + c\varepsilon^{\alpha-\frac{1}{2}}\ ;\ c,\mu > 0\ .$$

On utilise le même lemme pour prouver ce résultat.

Remarque 1.5 Pour $\alpha > 2$ on a obtenu une estimation valable pour des instants loin de l'instant initial. Neanmoins, on peut aussi obtenir une estimation valable près de l'instant initial:

$$E[|\hat{X}_k - M_k|] \leq c\exp\{-\mu\frac{t_k}{\varepsilon}\} + c\varepsilon\ ;\ c,\mu > 0\ , \tag{53}$$

puisque

$$\begin{aligned}
\sum_{i=1}^{k} E[|X_{i-1} - M_{i-1}||\phi_i|Z_{k-1,i-1}] \ &\leq\ c\rho(1+c\sqrt{\varepsilon})\sum_{i=0}^{k-1}(1+\sigma h\frac{\Delta t}{\varepsilon} - c_b\Delta t)^{-(k-i)} \\
&\quad +(1-\sigma h\frac{\Delta t}{\varepsilon} + c_b\Delta t)(1+\sigma h\frac{\Delta t}{\varepsilon} - c_b\Delta t)^{-(k-1)} \\
&\quad .\ E[|X_0 - M_0|]\ ,\ \text{car } 1 - A \leq 1 \\
&\leq\ c\rho(1+c\sqrt{\varepsilon})\frac{1+\sigma h\frac{\Delta t}{\varepsilon} - c_b\Delta t}{\sigma h\frac{\Delta t}{\varepsilon} - c_b\Delta t} \\
&\quad +(1-c\frac{\Delta t}{\varepsilon})^k E[|X_0 - M_0|] \\
&\leq\ c\exp\{-c\frac{t_k}{\varepsilon}\} + c\varepsilon\ .
\end{aligned}$$

Lemme 1.6 *:*

Soit $\alpha > 1$.

Soit F_k un processus adapté (dépendant de ε), différentiable par rapport à X_0 ($\forall k = 0,1,\cdots,K$).

Alors, si les moments de $\{\frac{\partial F}{\partial X_0}\}_k$ sont finis et si $F_k = O(\varepsilon^q)$, pour un certain $q \geq 0$,

$$E[F_k\frac{\partial}{\partial X_0}\log(L_k\Lambda_k)Z_{k-1,0}|Y_0^k] = -E[\frac{\partial F_k}{\partial X_0}Z_{k-1,0}|Y_0^k] + O(\varepsilon^{q+1})\ . \tag{54}$$

Pour $F_k \equiv 1$ ce résultat est plus fort que (51) e (52).

Preuve

On rappelle d'abbord que:

Si $g \in C^1$ et g et $\frac{\partial g}{\partial x}$ sont des fonction intégrables par rapport à la mesure de Lebesgue, alors

$$\int \frac{\partial g}{\partial x} dx = 0 \,. \tag{55}$$

En particulier,

Soit ψ une v.a. différentiable par rapport à X_0 telle que:

$$\tilde{E}[|\psi| + |\psi \frac{p_0'}{p_0}(X_0)| + |\frac{\partial \psi}{\partial X_0}|] < \infty \,. \tag{56}$$

Prenons une vèrsion $\psi(x, \tilde{w}_k, y_k)$ différentiable par rapport à x et posons

$$g(x) = p_0(x)\psi(x, \tilde{w}_k, y_k) \,.$$

Alors

$$\tilde{E}[\frac{\partial \psi}{\partial X_0} + \psi \frac{p_0'}{p_0}(X_0)|\tilde{w}, y] = 0 \,. \tag{57}$$

On applique la formule de changement de probabilités,

$$\begin{aligned}
E[F_k \frac{\partial}{\partial X_0} \log(L_k\Lambda_k) Z_{k-1,0}|Y_0^k] &= \frac{\tilde{E}[F_k \frac{\partial}{\partial X_0} \log(L_k\Lambda_k) Z_{k-1,0} L_k\Lambda_k|Y_0^k]}{\tilde{E}[L_k\Lambda_k|Y_0^k]} \\
&= \frac{\tilde{E}[F_k \frac{\frac{\partial}{\partial X_0}(L_k\Lambda_k)}{L_k\Lambda_k} Z_{k-1,0} L_k\Lambda_k|Y_0^k]}{\tilde{E}[L_k\Lambda_k|Y_0^k]} \\
&= \frac{\tilde{E}[F_k \frac{\partial}{\partial X_0}(L_k\Lambda_k) Z_{k-1,0}|Y_0^k]}{\tilde{E}[L_k\Lambda_k|Y_0^k]} \,.
\end{aligned}$$

Soit

$$\tilde{\phi}_k \triangleq \tilde{E}[F_k \frac{\partial}{\partial X_0}(L_k\Lambda_k) Z_{k-1,0}|Y_0^k] \,.$$

Puisque

$$\begin{aligned}\frac{\partial}{\partial X_0}(F_k(L_k\Lambda_k)Z_{k-1,0}) &= \frac{\partial F_k}{\partial X_0}(L_k\Lambda_k)Z_{k-1,0} + F_k\frac{\partial}{\partial X_0}(L_k\Lambda_k)Z_{k-1,0} \\ &\quad + F_k(L_k\Lambda)\frac{\partial}{\partial X_0}Z_{k-1,0}\end{aligned}$$

on a:

$$\begin{aligned}\tilde{\phi}_k &= \tilde{E}[\frac{\partial}{\partial X_0}(F_k(L_k\Lambda_k)Z_{k-1,0})|Y_0^k] - \tilde{E}[F_k(L_k\Lambda_k)\frac{\partial}{\partial X_0}Z_{k-1,0}|Y_0^k] \\ &\quad -\tilde{E}[\frac{\partial F_k}{\partial X_0}(L_k\Lambda_k)Z_{k-1,0}|Y_0^k]\,.\end{aligned}$$

Étant donné que $F_k(L_k\Lambda_k)Z_{k-1,0}$ vérifie la condition d'intégrabilité (56) on applique (57) pour obtenir:

$$\tilde{E}[\frac{\partial}{\partial X_0}(F_k(L_k\Lambda_k)Z_{k-1,0}|Y_0^k] = -\tilde{E}[F_k(L_k\Lambda_k)Z_{k-1,0}\frac{p_0'}{p_0}(X_0)|Y_0^k]$$

et alors

$$\begin{aligned}\tilde{\phi}_k &= -\tilde{E}[\frac{p_0'}{p_0}(X_0)F_k(L_k\Lambda_k)Z_{k-1,0}|Y_0^k] - \tilde{E}[F_k(L_k\Lambda_k)\frac{\partial}{\partial X_0}Z_{k-1,0}|Y_0^k] \\ &\quad -\tilde{E}[\frac{\partial F_k}{\partial X_0}(L_k\Lambda_k)Z_{k-1,0}|Y_0^k]\,.\end{aligned}$$

Soit

$$\begin{aligned}\phi_k &\triangleq E[F_k\frac{\partial}{\partial X_0}\log(L_k\Lambda_k)Z_{k-1,0}|Y_0^k] \\ &= -\frac{\tilde{E}[(L_k\Lambda_k)\frac{p_0'}{p_0}(X_0)F_kZ_{k-1,0}|Y_0^k]}{\tilde{E}[L_k\Lambda_k|Y_0^k]} - \frac{\tilde{E}[(L_k\Lambda_k)F_k\frac{\partial}{\partial X_0}Z_{k-1,0}|Y_0^k]}{\tilde{E}[L_k\Lambda_k|Y_0^k]} \\ &\quad -\frac{\tilde{E}[(L_k\Lambda_k)\frac{\partial F_k}{\partial X_0}Z_{k-1,0}|Y_0^k]}{\tilde{E}[L_k\Lambda_k|Y_0^k]} \\ &= -E[\frac{p_0'}{p_0}(X_0)F_kZ_{k-1,0}|Y_0^k] - E[F_k\frac{\partial}{\partial X_0}Z_{k-1,0}|Y_0^k] \\ &\quad -E[\frac{\partial F_k}{\partial X_0}Z_{k-1,0}|Y_0^k]\,. \qquad (58)\end{aligned}$$

Puisqu'on avait supposé qué:

$$\int |\frac{p_0'}{p_0}(x)|^r p_0(x)dx < \infty \ , \ p_0 \in C^1 \ . \tag{59}$$

e, de (45),

$$\begin{aligned} Z_{K-1,0} &\leq (1+\sigma h\frac{\Delta t}{\varepsilon} - c_b\Delta t)^{-(k-1)} \\ &\leq c\exp\{-c(k-1)\frac{\frac{\Delta t}{\varepsilon}}{1+c\frac{\Delta t}{\varepsilon}}\} \\ &\leq c\exp\{-c\frac{t_{k-1}}{\varepsilon}\} \end{aligned}$$

on a que le 1^{ier} tèrme est d'ordre $c\varepsilon^q \exp\{-c\frac{t_{k-1}}{\varepsilon}\}$ et le $3^{ième}$ tèrme est le membre à droîte dans l'expression (54). Alors, il nous suffit de demontrer que:

$$E[F_k\frac{\partial}{\partial X_0}Z_{k-1,0}|Y_0^k] = O(\varepsilon^{q+1}) \ . \tag{60}$$

Or, de (44), on a que:

$$Z_{k-1,0} = Z_{k-2,0}(1+\sigma h\frac{\Delta t}{\varepsilon} + \Delta t b'(X_{k-1}))^{-1}$$

donc, par indution, on obtient la formule générale pour la dérivée partielle par rapport à la condition initiale:

$$\begin{aligned} \frac{\partial}{\partial X_0}Z_{n0} &= -\Delta t\sum_{i=0}^{n-1} b''(X_i)\frac{\partial X_i}{\partial X_0}(1+\sigma h\frac{\Delta t}{\varepsilon} + \Delta t b'(X_i))^{-1} \\ &\quad \cdot \prod_{j=0}^{n-1}(1+\sigma h\frac{\Delta t}{\varepsilon} + \Delta t b'(X_j))^{-1} \\ &= -\Delta t\sum_{i=0}^{n-1}\frac{b''(X_i)}{Z_{i0}}(1+\sigma h\frac{\Delta t}{\varepsilon} + \Delta t b'(X_i))^{-1} \\ &\quad \cdot \prod_{j=0}^{n-1}(1+\sigma h\frac{\Delta t}{\varepsilon} + \Delta t b'(X_j))^{-1} \ , \\ &\quad \text{puisque } \frac{\partial X_i}{\partial X_0} = Z_{0i} = \frac{1}{Z_{i0}} \\ &= -\Delta t\sum_{i=0}^{n-1} b''(X_i)\left(\prod_{j=0}^{i-1}(1+\sigma h\frac{\Delta t}{\varepsilon} + \Delta t b'(X_j))\right)(1+\sigma h\frac{\Delta t}{\varepsilon} + \Delta t b'(X_i))^{-1} \end{aligned}$$

$$
\begin{aligned}
& \cdot \left(\prod_{j=0}^{n-1} (1 + \sigma h \frac{\Delta t}{\varepsilon} + \Delta t b'(X_j))^{-1} \right) \\
= & -\Delta t \sum_{i=0}^{n-1} b''(X_i)(1 + \sigma h \frac{\Delta t}{\varepsilon} + \Delta t b'(X_i))^{-1} \\
& \cdot \prod_{j=i}^{n-1} (1 + \sigma h \frac{\Delta t}{\varepsilon} + \Delta t b'(X_j))^{-1}
\end{aligned}
$$

et

$$
\frac{\partial}{\partial X_0} Z_{0,0} = 0
$$

donc

$$
\begin{aligned}
|\frac{\partial}{\partial X_0} Z_{n0}| & \leq \Delta t \sum_{i=0}^{n-1} |b''(X_i)| (1 + \sigma h \frac{\Delta t}{\varepsilon} - c_b \Delta t)^{-(n-i+1)} \\
& \leq \Delta t ||b''|| (1 + \sigma h \frac{\Delta t}{\varepsilon} - c_b \Delta t)^{-1} \sum_{i=1}^{n} (1 + \sigma h \frac{\Delta t}{\varepsilon} - c_b \Delta t)^{-i} \\
& \leq \Delta t ||b''|| (1 + \sigma h \frac{\Delta t}{\varepsilon} - c_b \Delta t)^{-1} \frac{(1 + \sigma h \frac{\Delta t}{\varepsilon} - c_b \Delta t)^{-1}}{1 - \frac{1}{1 + \sigma h \frac{\Delta t}{\varepsilon} - c_b \Delta t}} \\
& = \Delta t ||b''|| (1 + \sigma h \frac{\Delta t}{\varepsilon} - c_b \Delta t)^{-1} \frac{1}{\sigma h \frac{\Delta t}{\varepsilon} - c_b \Delta t} \\
& \leq \Delta t ||b''|| \frac{1}{\sigma h \frac{\Delta t}{\varepsilon} - c_b \Delta t} = O(\varepsilon) ,
\end{aligned}
$$

ce qui prouve (60) et tèrmine la démonstration du lemme.

■

• Enfin, pour obtenir une estimation de l'erreur comise dans une étape de filtrage, il nous suffit de remarquer qu'une approximation de $\tilde{X}_k = E[X_{k-1}|Y_0^k]$ est donnée par le schéma (35), étant

$$
\hat{X}_k - \tilde{X}_k = E[X_k|Y_0^k] - E[X_{k-1}|Y_0^k]
$$

$$
\begin{aligned}
&= E[(b(X_{k-1})\Delta t + \sigma\sqrt{\Delta t}w_k|Y_0^k] \\
&= \Delta t E[b(X_{k-1})|Y_0^k]
\end{aligned}
$$

i.e.

$$|\hat{X}_k - \tilde{X}_k| \leq c\Delta t\,. \tag{61}$$

Références

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A Version discrète du théorème de Girsanov

Considèrons une filtration $(\mathcal{F}_k)_k$, $k \in \{0, 1, \cdots K\}$.
Soient:

- $\{w_k\}$ un $\mathcal{F}_k$-bruit blanc gaussien pour la probabilité P.
- $\{\varphi_k\}$ un processus $\mathcal{F}_{k-1}$-mesurable (i.e. prévisible) tel que:

$$\sum_{i=0}^{K} |\varphi_i|^2 < \infty \qquad (P\ p.s.)$$

- $\{Z_k\}$ le processus définit par:

$$\begin{aligned} Z_k &= \exp\{\textstyle\sum_{i=1}^k \varphi_i w_i - \frac{1}{2}\sum_{i=1}^k |\varphi_i|^2\} \ , \ k \geq 1 \\ Z_0 &= 1 \ . \end{aligned}$$

Alors,

1. $\{Z_k\}$ **est une martingale discrète.**

Preuve

$$\begin{aligned} E[Z_k|\mathcal{F}_{k-1}] &= E[Z_{k-1}\frac{Z_k}{Z_{k-1}}|\mathcal{F}_{k-1}] \\ &= Z_{k-1}E[\exp\{\varphi_k w_k - \frac{1}{2}|\varphi_k|^2\}|\mathcal{F}_{k-1}] \ , \\ &\quad \text{puisque } Z_{k-1} \text{ est } \mathcal{F}_{k-1}\text{-mesurable.} \\ &= Z_{k-1} \ , \\ &\quad \text{puisque } \{w_k\} \text{ est un b.b. gaussien et } \varphi_k \text{ est } \mathcal{F}_{k-1}\text{-mesurable.} \end{aligned}$$

et Z_k est $\mathcal{F}_k$ intégrable.

∎

On considère la probabilité $\bar{P}$ définie par:

$$d\bar{P}(\omega) = Z_K(\omega)dP(\omega) \ . \tag{62}$$

Remarque A.1

Du fait que $\{Z_k\}$ est une martingale, vient que:

$$\frac{d\bar{P}}{dP}\Big|_{\mathcal{F}_k} = Z_k \ .$$

2. **(Vèrsion discrète du Théorème de Girsanov)**

Pour la probabilité $\bar{P}$ définie dans (62), le processus $\{\bar{w}_k\}$ défini par:

$$\bar{w}_k = w_k - \varphi_k$$

est un b.b. gaussien .

Preuve

On veut démontrer que:

$$\bar{E}[\exp(\lambda \bar{w}_k)|\mathcal{F}_{k-1}] = \exp(\frac{\lambda^2}{2}) , \quad \forall \lambda \in \mathbb{R}$$

i.e.

$$\bar{E}[\exp(\lambda \bar{w}_k - \frac{\lambda^2}{2})|\mathcal{F}_{k-1}] = 1 , \quad \forall \lambda \in \mathbb{R} .$$

Or,

$$\begin{aligned}
\bar{E}[\exp(\lambda \bar{w}_k - \frac{\lambda^2}{2})|\mathcal{F}_{k-1}] &= \frac{E[\exp(\lambda \bar{w}_k - \frac{\lambda^2}{2})Z_K|\mathcal{F}_{k-1}]}{E[Z_K|\mathcal{F}_{k-1}]} \quad \text{(formule de Beyes)} \\
&= \frac{E[\exp(\lambda w_k - \lambda\varphi_k - \frac{\lambda^2}{2})Z_k|\mathcal{F}_{k-1}]}{Z_{k-1}} ,
\end{aligned}$$

puisque Z_k est une martingale et d'après la remarque ci dessus.

$$\begin{aligned}
&= E[\exp(\lambda w_k - \lambda\varphi_k - \frac{\lambda^2}{2}) \exp(\varphi_k w_k - \frac{1}{2}\varphi_k^2)|\mathcal{F}_{k-1}] \\
&= E\{\exp[(\lambda + \varphi_k)w_k - \frac{1}{2}(\lambda + \varphi_k)^2]|\mathcal{F}_{k-1}\} \\
&= 1 \text{ , de même qu'en 1. .}
\end{aligned}$$

■

REFINED AND HIGH–ORDER TIME–DISCRETIZATION OF NONLINEAR FILTERING EQUATIONS*

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preliminary version

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Contents

1 Introduction

The purpose of this paper is to provide numerical approximation schemes for the following abstract stochastic differential equation

$$du_t + Au_t\,dt = \sum_{i=1}^{d} B_i u_t\,dY_t^i\ ,$$
$$u_0 = \bar{u}\ , \tag{1.1}$$

where the operator A is unbounded in some separable Hilbert space H. On the other hand, the operators $(B_1, B_2, \cdots, B_d)$ are supposed bounded. The most important example of equation of this type is provided by Zakaï equation of nonlinear filtering.

Similarly to the deterministic case, it is possible to associate a semi-group (actually a stochastic semi-group, according to Skorokhod), with equation (1.1). The approach adopted here is to build approximations of the stochastic differential equation as approximations of the corresponding semi-group.

In Section 2, basic definition and properties of random linear operators and stochastic semi-groups are presented, following Skorokhod. A general approximation theorem for stochastic semi-groups is proved, with error estimates. This theorem can be thought of as an extension of Theorem 2.2 in Newton [6, p.32], based itself on earlier work by Wagner and Platen [9,10], to stochastic semi-groups. In Section 3, the existence and uniqueness theorem of [7] is completed with an abstract regularity result. In addition, the semi-group associated with equation (1.1) is defined. In Section 4, the following time-discretization scheme is investigated

$$\bar{u}_{n+1} = \left[P_{t_{n+1}-t_n}\ \Psi_{t_{n+1}}^{t_n}\right]\ \bar{u}_n$$

$$\bar{u}_0 = \bar{u}$$

where $(P_t\ :\ t \geq 0)$ is the (deterministic) semi-group generated by $-A$, whereas the two-parameter semi-group $(\Psi_t^s\ :\ 0 \leq s \leq t)$ is defined by

$$\Psi_t^s \triangleq \exp\left[\sum_{i=1}^{d} B_i\,(Y_t^i - Y_s^i) - \frac{1}{2}\sum_{i=1}^{d} B_i^2\,(t-s)\right]\ .$$

This is nothing but a Trotter-like product formula, with the attractive feature that the deterministic and the stochastic part are decoupled. It is proved using the approximation theorem of Section 2, that $\overline{u}_n$ approximates u_{t_n} in the L^2-sense, and that the speed of convergence is of order $O(k)$, where k denotes the time-step. In addition, in the context of nonlinear filtering, it is possible to give a simple probabilistic interpretation to this time-discretization scheme, following the approch of [3].

2 Stochastic semi-groups

In the next two subsections, definitions are given concerning random linear operators and stochastic semi-groups, following the work of Skorokhod [11,12]. If not explicitely stated, all the vector spaces to be considered here are separable Hilbert spaces. Let $(\Omega, \mathcal{F}, \mathrm{P})$ be the underlying probability space.

2.1 Random linear operators

Definition 2.1 *A (strong) random linear operator from F into G is a linear and continuous operator from F into $L^2(\Omega, \mathcal{F}\,; G)$. The set of all such operators will be denoted by $\mathcal{L}^2_s(F, G)$.*

Remark. Let $U \in \mathcal{L}^2_s(F,G)$. Then only Ux, for $x \in F$, is defined as a G-valued random variable. In particular, U itself is not a random variable taking values in $\mathcal{L}(F,G)$. However

(i) The mapping: $x \mapsto \mathbf{E}(Ux)$ defines a linear and continuous operator from F into G, which will be denoted by $\mathbf{E}(U)$, in the following way

$$\forall x \in F\,, \quad \mathbf{E}(U)x \triangleq \mathbf{E}(Ux)\,. \tag{2.1}$$

(ii) In the same way, the mapping: $(x,y) \mapsto \mathbf{E}(Ux, Uy)_G$ is a symmetric and continuous bilinear form on $F \times F$, which uniquely defines a linear and continuous self-adjoint operator on F, which will be denoted by $\mathbf{E}(U^*U)$, in the following way

$$\forall x, y \in F\,, \quad (\mathbf{E}(U^*U)x, y)_F \triangleq \mathbf{E}(Ux, Uy)_G\,. \tag{2.2}$$

In particular, this allows to define the following norm in $\mathcal{L}^2_s(F,G)$

$$\|U\|_{\mathcal{L}^2_s(F,G)} \triangleq \|\mathbf{E}(U^*U)\|^{1/2}_{\mathcal{L}(F,F)}\,. \tag{2.3}$$

(iii) More generally, let $U_i \in \mathcal{L}^2_s(F_i, G_i)$ $(i = 1,2)$. For all $C \in \mathcal{L}(G_1, G_2)$, the mapping: $(x,y) \mapsto \mathbf{E}(CU_1x, U_2y)_{G_2}$ is a continuous bilinear form on $F_1 \times F_2$, which uniquely defines a linear and continuous operator from F_1 to F_2, which will be denoted by $\mathbf{E}(U_2^*CU_1)$, in the following way

$$\forall x \in F_1\,, \forall y \in F_2\,, \quad (\mathbf{E}(U_2^*CU_1)x, y)_{F_2} \triangleq \mathbf{E}(CU_1x, U_2y)_{G_2}\,. \tag{2.4}$$

Proposition 2.2 *The (deterministic) operator Θ_{U_1,U_2} defined by*

$$\forall C \in \mathcal{L}(G_1, G_2)\,, \quad \Theta_{U_1,U_2}(C) \triangleq \mathbf{E}(U_2^*CU_1)\,, \tag{2.5}$$

is linear and continuous from $\mathcal{L}(G_1, G_2)$ into $\mathcal{L}(F_1, F_2)$.

PROOF. By definition

$$
\begin{aligned}
|(\Theta_{U_1,U_2}(C)x,y)_{F_2}| &\le \left(\mathbf{E}|CU_1x|^2_{G_1}\right)^{1/2} \left(\mathbf{E}|U_2y|^2_{G_2}\right)^{1/2} \\
&\le \|C\|_{\mathcal{L}(G_1,G_2)} \left(\mathbf{E}|U_1x|^2_{G_1}\right)^{1/2} \left(\mathbf{E}|U_2y|^2_{G_2}\right)^{1/2} \\
&\le \|C\|_{\mathcal{L}(G_1,G_2)} \|U_1\|_{\mathcal{L}^2_s(F_1,G_1)} \|U_2\|_{\mathcal{L}^2_s(F_2,G_2)} |x|_{F_1} |y|_{F_2} ,
\end{aligned}
$$

from which it follows that

$$
\|\Theta_{U_1,U_2}(C)\|_{\mathcal{L}(F_1,F_2)} \le \|C\|_{\mathcal{L}(G_1,G_2)} \|U_1\|_{\mathcal{L}^2_s(F_1,G_1)} \|U_2\|_{\mathcal{L}^2_s(F_2,G_2)} , \tag{2.6}
$$

which proves the assertion. □

Remark. It might happen that the estimate (2.6) is not tight enough. Then it can be noted that $\forall x \in F_1$, $\forall y \in F_2$

$$
\begin{aligned}
(\Theta_{U_1,U_2}(C)x,y)_{F_2} &= \mathbf{E}(CU_1x, U_2y)_{G_2} \\
&= (C\mathbf{E}(U_1)x, \mathbf{E}(U_2)y)_{G_2} + \mathbf{E}(C(U_1 - \mathbf{E}(U_1))x, (U_2 - \mathbf{E}(U_2))y)_{G_2} ,
\end{aligned}
$$

which gives

$$
\begin{aligned}
\|\Theta_{U_1,U_2}(C)\|_{\mathcal{L}(F_1,F_2)} \le \|C\|_{\mathcal{L}(G_1,G_2)} &\{\|\mathbf{E}(U_1)\|_{\mathcal{L}(F_1,G_1)} \|\mathbf{E}(U_2)\|_{\mathcal{L}(F_2,G_2)} \\
&+ \|U_1 - \mathbf{E}(U_1)\|_{\mathcal{L}^2_s(F_1,G_1)} \|U_2 - \mathbf{E}(U_2)\|_{\mathcal{L}^2_s(F_2,G_2)}\} .
\end{aligned} \tag{2.7}
$$

Definition 2.3 *Let $\mathcal{B} \subset \mathcal{F}$ be a σ-algebra. A random linear operator $U \in \mathcal{L}^2_s(F,G)$ is $\mathcal{B}$-measurable if and only if $\forall x \in F$ the G-valued random variable Ux is $\mathcal{B}$-measurable.*

With this definition, it is possible to apply a random linear operator to a random vector, provided they are mutually independent. Indeed

Proposition 2.4 *Let $U \in \mathcal{L}^2_s(F,G)$ and $\mathcal{A}, \mathcal{B} \subset \mathcal{F}$ be two mutually independent σ-algebras. If U is $\mathcal{B}$-measurable, then it can be extended as a linear and continuous operator from $L^2(\Omega,\mathcal{A}\,;F)$ into $L^2(\Omega,\mathcal{A}\vee\mathcal{B}\,;G)$, with same norm.*

Remark. In addition, the mapping: $x \mapsto \mathbf{E}(Ux|\mathcal{A})$ defines a linear and continuous operator from $L^2(\Omega,\mathcal{A}\,;F)$ into $L^2(\Omega,\mathcal{A}\,;G)$, which coincides with $\mathbf{E}(U)$.

PROOF. First, let $x \in L^2(\Omega,\mathcal{A}\,;F)$ be simple, i.e.

$$
x = \sum_{i=1}^{n} a_i \, 1_{A_i} , \quad \text{with } a_i \in F \text{ and } A_i \in \mathcal{A} .
$$

It is natural to set

$$Ux \triangleq \sum_{i=1}^{n} Ua_i \, 1_{A_i} \ ,$$

which gives

$$\begin{aligned}
\mathbf{E}(|Ux|_G^2) &= \sum_{i=1}^{n} \mathbf{E}(|Ua_i|_G^2 \, 1_{A_i}) \\
&= \sum_{i=1}^{n} \mathbf{E}(|Ua_i|_G^2) \, \mathrm{P}(A_i) \\
&\leq \|U\|^2_{\mathcal{L}^2_s(F,G)} \sum_{i=1}^{n} |a_i|_F^2 \, \mathrm{P}(A_i) = \|U\|^2_{\mathcal{L}^2_s(F,G)} \, \mathbf{E}|x|_F^2 \ .
\end{aligned}$$

The same inequality holds for all $x \in L^2(\Omega, \mathcal{A}\,;\, F)$, by density of simple random variables.

For the reverse inequality, it is enough to remark that

$$\sup_{0 \neq x \in L^2(\Omega,\mathcal{A}\,;\,F)} \frac{\mathbf{E}|Ux|_G^2}{\mathbf{E}|x|_F^2} \geq \sup_{0 \neq x \in F} \frac{\mathbf{E}|Ux|_G^2}{|x|_F^2} = \|U\|^2_{\mathcal{L}^2_s(F,G)} \ .$$

□

Proposition 2.4 allows to define the product of mutually independent random linear operators. Indeed, let $U \in \mathcal{L}^2_s(F,G)$ and $V \in \mathcal{L}^2_s(G,H)$. If U and V are mutually independent, then the product operator $V\,U$ can be defined as an element of $\mathcal{L}^2_s(F,H)$. Moreover

$$\|V\,U\|_{\mathcal{L}^2_s(F,H)} \leq \|U\|_{\mathcal{L}^2_s(F,G)} \, \|V\|_{\mathcal{L}^2_s(G,H)} \ .$$

The purpose of the next proposition is to prove a morphism property for the mapping: $(U_1, U_2) \mapsto \Theta_{U_1,U_2}$ defined by (2.4) and (2.5).

Proposition 2.5 *Let $U_i \in \mathcal{L}^2_s(F_i, G_i)$ and $V_i \in \mathcal{L}^2_s(G_i, H_i)$ $(i = 1, 2)$. Assume that (U_1, U_2) and (V_1, V_2) are mutually independent. Then*

$$\Theta_{V_1 U_1, V_2 U_2} = \Theta_{U_1,U_2} \circ \Theta_{V_1,V_2} \ . \tag{2.8}$$

PROOF. Let $C \in \mathcal{L}(H_1, H_2)$. By definition, $\forall x \in F_1$, $\forall y \in F_2$

$$\begin{aligned}
(\Theta_{V_1 U_1, V_2 U_2}(C)x, y)_{F_2} &= \mathbf{E}(CV_1U_1x, V_2U_2y)_{H_2} \\
&= \mathbf{E}(\mathbf{E}(V_2^* C V_1) U_1 x, U_2 y)_{G_2} \\
&= (\mathbf{E}(U_2^* \Theta_{V_1,V_2}(C) U_1) x, y)_{F_2} \\
&= (\Theta_{U_1,U_2}(\Theta_{V_1,V_2}(C))\, x, y)_{F_2} \ .
\end{aligned}$$

□

2.2 Stochastic semi-groups

Definition 2.6 *A (strong) stochastic semi-group in H is a two-parameter family $(U_t^s : 0 \leq s \leq t)$ of (strong) random linear operators in H, satisfying*

(i) for all $s \leq t \leq u \leq v$, U_t^s and U_v^u are mutually independent,

(ii) for all $s \leq t \leq u$, $\quad U_u^s = U_u^t\, U_t^s$.

The stochastic semi-group is strongly continuous if

(iii) $\forall x \in H \; ; \quad \lim_{k \to 0^+} \mathbf{E}|U_{t+k}^t x - x|_H^2 = 0$.

Remark. The independence property *(i)* makes it possible to define the product operators appearing in the semi-group property *(ii)*.

The next Proposition gives a sufficient condition for the independence hypothesis *(i)* to hold. Indeed

Proposition 2.7 *If for all $s \leq t$, U_t^s is $\mathcal{Y}_t^s$-measurable, where the two-parameter family $(\mathcal{Y}_t^s : 0 \leq s \leq t)$ of σ-algebras satisfies*

(iv) for all $s \leq t \leq u \leq v$,

$$\mathcal{Y}_t^s \text{ and } \mathcal{Y}_v^u \text{ are mutually independent,}$$

$$\mathcal{Y}_t^s \vee \mathcal{Y}_v^u \subset \mathcal{Y}_v^s \; ,$$

then (i) holds.

Proposition 2.8 *The two-parameter family $(Q_t^s : 0 \leq s \leq t)$ of bounded linear (deterministic) operators in H, defined by $Q_t^s \triangleq \mathbf{E}(U_t^s)$, is a non-homogeneous strongly continuous semi-group in H.*

Definition 2.9 *A discrete stochastic semi-group in H is given by a family $(U_{i+1}^i : i = 0, 1, \cdots)$ of random linear operators in H, such that for all $j < i$, U_{j+1}^j and U_{i+1}^i are mutually independent. The semi-group itself is the two-parameter family $(U_m^l : 0 \leq l \leq m)$ defined by*

$$U_m^l \triangleq \prod_{i=l}^{m-1} U_{i+1}^i \; .$$

By convention $U_m \triangleq U_m^0$.

2.3 Approximation of stochastic semi–groups

The next Theorem is an extension of Theorem 2.2 in Newton [6, p.32], based itself on earlier work by Wagner and Platen [9,10], to stochastic semi–groups.

Roughly speaking, this result says that, if the one–step error between two stochastic semi–groups is of order $O(k^{(r+1)/2})$, then the overall error will be of order $O(k^{r/2})$, provided the expected value of the one–step error is of order $O(k^{r/2+1})$, where the estimations are understood in the L^2–sense.

To be specific, let $\pi\,:\;0=t_0<t_1<\cdots<t_i<\cdots<t_n=T$ be a partition of the interval $[0,T]$, with mesh–size k.

Theorem 2.10 *Let $(U^l_m\ ;\ 0\le l\le m)$ and $(V^l_m\ ;\ 0\le l\le m)$ be two discrete stochastic semi–groups in H. Suppose that $U^i_{i+1}\in\mathcal{L}^2_s(D,D)$ where $D\subset H$.*

Suppose that the following stability estimates hold for the two discrete stochastic semi–groups

$$\|U^i_{i+1}\|_{\mathcal{L}^2_s(D,D)}\ \le\ e^{\frac{1}{2}\beta^2_D(t_{i+1}-t_i)}\ , \tag{2.9}$$

$$\|\mathbf{E}(U^i_{i+1})\|_{\mathcal{L}(D,D)}\ \le\ 1\ , \tag{2.10}$$

and

$$\|V^i_{i+1}\|_{\mathcal{L}^2_s(H,H)}\ \le\ e^{\frac{1}{2}\gamma^2_H(t_{i+1}-t_i)}\ , \tag{2.11}$$

$$\|\mathbf{E}(V^i_{i+1})\|_{\mathcal{L}(H,H)}\ \le\ 1\ . \tag{2.12}$$

If the following consistency estimates hold for the one–step error $\delta^i_{i+1}\triangleq U^i_{i+1}-V^i_{i+1}$

$$\|\delta^i_{i+1}\|_{\mathcal{L}^2_s(D,H)}\ \le\ \alpha_0(t_{i+1}-t_i)^{(r+1)/2}\ , \tag{2.13}$$

$$\|\mathbf{E}(\delta^i_{i+1})\|_{\mathcal{L}(D,H)}\ \le\ \alpha_1(t_{i+1}-t_i)^{r/2+1}\ , \tag{2.14}$$

then the overall error satisfies

$$\|U_n-V_n\|_{\mathcal{L}^2_s(D,H)}=O(k^{r/2})\ .$$

PROOF. From the decomposition

$$U_n-V_n=\sum_{i=0}^{n-1}V^{i+1}_n\,(U^i_{i+1}-V^i_{i+1})\,U_i\ ,$$

it follows, introducing $\Delta_i\triangleq V^{i+1}_n\,(U^i_{i+1}-V^i_{i+1})\,U_i$ that

$$\forall\bar u\in H\,,\quad \mathbf{E}|U_n\bar u-V_n\bar u|^2_H=\sum_{i=0}^{n-1}\mathbf{E}|\Delta_i\bar u|^2_H+2\sum_{i=0}^{n-1}\sum_{j=0}^{i-1}\mathbf{E}(\Delta_i\bar u,\Delta_j\bar u)_H\ . \tag{2.15}$$

The rest of the proof is divided in two parts.

□ *Analysis of square terms*

By Proposition 2.5, for all $\bar{u} \in D$

$$\begin{aligned} \mathbf{E}|\Delta_i \bar{u}|_H^2 &= \mathbf{E}(V_n^{i+1}\delta_{i+1}^i U_i \bar{u}, V_n^{i+1}\delta_{i+1}^i U_i \bar{u})_H \\ &= (\Theta_{U_i,U_i} \circ \Theta_{\delta_{i+1}^i,\delta_{i+1}^i} \circ \Theta_{V_n^{i+1},V_n^{i+1}}(I)\bar{u}, \bar{u})_D \ . \end{aligned}$$

Each of these three components will be studied separately, making use of Proposition 2.2 and estimate (2.6).

- $V_n^{i+1} \in \mathcal{L}_s^2(H,H)$, therefore for all $C \in \mathcal{L}(H,H)$

  $$\Theta_{V_n^{i+1},V_n^{i+1}}(C) \in \mathcal{L}(H,H) \ ,$$

  and

  $$\|\Theta_{V_n^{i+1},V_n^{i+1}}(C)\|_{\mathcal{L}(H,H)} \le \|C\|_{\mathcal{L}(H,H)} \, \|V_n^{i+1}\|_{\mathcal{L}_s^2(H,H)}^2 \le \|C\|_{\mathcal{L}(H,H)} \, e^{\gamma_H^2(T-t_i)} \ .$$

- $\delta_{i+1}^i \in \mathcal{L}_s^2(D,H)$, therefore for all $C \in \mathcal{L}(H,H)$

  $$\Theta_{\delta_{i+1}^i,\delta_{i+1}^i}(C) \in \mathcal{L}(D,D) \ ,$$

  and

  $$\|\Theta_{\delta_{i+1}^i,\delta_{i+1}^i}(C)\|_{\mathcal{L}(D,D)} \le \|C\|_{\mathcal{L}(H,H)} \, \|\delta_{i+1}^i\|_{\mathcal{L}_s^2(D,H)}^2 \le \|C\|_{\mathcal{L}(H,H)} \, \alpha_0^2(t_{i+1} - t_i)^{r+1} \ .$$

- $U_i \in \mathcal{L}_s^2(D,D)$, therefore for all $C \in \mathcal{L}(D,D)$

  $$\Theta_{U_i,U_i}(C) \in \mathcal{L}(D,D) \ ,$$

  and

  $$\|\Theta_{U_i,U_i}(C)\|_{\mathcal{L}(D,D)} \le \|C\|_{\mathcal{L}(D,D)} \, \|U_i\|_{\mathcal{L}_s^2(D,D)}^2 \le \|C\|_{\mathcal{L}(D,D)} \, e^{\beta_D^2 t_i} \ .$$

It follows from this first analysis that for all $\bar{u} \in D$

$$\begin{aligned} \mathbf{E}|\Delta_i \bar{u}|_H^2 &\le \alpha_0^2(t_{i+1} - t_i)^{r+1} \, e^{\gamma_H^2(T-t_{i+1})} \, e^{\beta_D^2 t_i} \, |\bar{u}|_D^2 \\ &\le \alpha_0^2 k^{r+1} \, e^{c^2 T} \, |\bar{u}|_D^2 \ , \end{aligned}$$

where $c \triangleq \max(\beta_D, \gamma_H)$.

□ ***Analysis of double-product terms***

Consider the following partition of the interval $[0,T]$ build from the original partition π, under the assumption that $j < i$

$$0 = t_0 \qquad t_j \quad t_{j+1} \qquad t_i \quad t_{i+1} \qquad t_n = T$$

Using this new partition

$$\Delta_i = V_n^{i+1}\, \delta_{i+1}^i\, U_i^{j+1}\, U_{j+1}^j\, U_j \;,$$

$$\Delta_j = V_n^{i+1}\, V_{i+1}^i\, V_i^{j+1}\, \delta_{j+1}^j\, U_j \;.$$

Therefore, by Proposition 2.5, for all $\bar{u} \in D$

$$\begin{aligned}\mathbf{E}(\Delta_i \bar{u}, \Delta_j \bar{u})_H &= \mathbf{E}(V_n^{i+1}\delta_{i+1}^i U_i^{j+1} U_{j+1}^j U_j \bar{u}, V_n^{i+1} V_{i+1}^i V_i^{j+1} \delta_{j+1}^j U_j \bar{u})_H \\ &= (\Theta_{U_j,U_j} \circ \Theta_{U_{j+1}^j, \delta_{j+1}^j} \circ \Theta_{U_i^{j+1}, V_i^{j+1}} \circ \Theta_{\delta_{i+1}^i, V_{i+1}^i} \circ \Theta_{V_n^{i+1}, V_n^{i+1}}(I)\bar{u}, \bar{u})_D \;.\end{aligned}$$

Each of these five components will be studied separately, making use of Proposition 2.2 and estimates (2.6), (2.7).

- $V_n^{i+1} \in \mathcal{L}_s^2(H,H)$, therefore for all $C \in \mathcal{L}(H,H)$

  $$\Theta_{V_n^{i+1}, V_n^{i+1}}(C) \in \mathcal{L}(H,H) \;,$$

  and

  $$\|\Theta_{V_n^{i+1}, V_n^{i+1}}(C)\|_{\mathcal{L}(H,H)} \le \|C\|_{\mathcal{L}(H,H)}\, \|V_n^{i+1}\|^2_{\mathcal{L}_s^2(H,H)} \le \|C\|_{\mathcal{L}(H,H)}\, e^{\gamma_H^2 (T - t_{i+1})} \;.$$

- $\delta_{i+1}^i \in \mathcal{L}_s^2(D,H)$ and $V_{i+1}^i \in \mathcal{L}_s^2(H,H)$, therefore for all $C \in \mathcal{L}(H,H)$

  $$\Theta_{\delta_{i+1}^i, V_{i+1}^i}(C) \in \mathcal{L}(D,H) \;,$$

  and

  $$\begin{aligned}\|\Theta_{\delta_{i+1}^i, V_{i+1}^i}(C)\|_{\mathcal{L}(D,H)} &\le \|C\|_{\mathcal{L}(H,H)} \{\|\mathbf{E}(\delta_{i+1}^i)\|_{\mathcal{L}(D,H)}\, \|\mathbf{E}(V_{i+1}^i)\|_{\mathcal{L}(D,H)} \\ &\qquad + \|\delta_{i+1}^i\|_{\mathcal{L}_s^2(D,H)}\, \|V_{i+1}^i - \mathbf{E}(V_{i+1}^i)\|_{\mathcal{L}_s^2(H,H)}\} \\ &\le \|C\|_{\mathcal{L}(H,H)} \{\alpha_1 (t_{i+1} - t_i)^{r/2+1} \\ &\qquad + \alpha_0 (t_{i+1} - t_i)^{(r+1)/2} (e^{\gamma_H^2 (t_{i+1} - t_i)} - 1)^{1/2}\} \\ &\le \|C\|_{\mathcal{L}(H,H)} (t_{i+1} - t_i)^{r/2+1} \{\alpha_1 + \alpha_0 \gamma_H e^{\frac{1}{2}\gamma_H^2 (t_{i+1} - t_i)}\} \;.\end{aligned}$$

- $U_i^{j+1} \in \mathcal{L}_s^2(D,D)$ and $V_i^{j+1} \in \mathcal{L}_s^2(H,H)$, therefore for all $C \in \mathcal{L}(D,H)$

  $\Theta_{U_i^{j+1},V_i^{j+1}}(C) \in \mathcal{L}(D,H)$,

  and

$$\begin{aligned}\|\Theta_{U_i^{j+1},V_i^{j+1}}(C)\|_{\mathcal{L}(D,H)} &\leq \|C\|_{\mathcal{L}(D,H)}\,\|U_i^{j+1}\|_{\mathcal{L}_s^2(D,D)}\,\|V_i^{j+1}\|_{\mathcal{L}_s^2(H,H)} \\ &\leq \|C\|_{\mathcal{L}(D,H)}\, e^{\frac{1}{2}(\beta_D^2+\gamma_H^2)(t_i-t_{j+1})} .\end{aligned}$$

- $U_{j+1}^j \in \mathcal{L}_s^2(D,D)$ and $\delta_{j+1}^j \in \mathcal{L}_s^2(D,H)$, therefore for all $C \in \mathcal{L}(D,H)$

  $\Theta_{U_{j+1}^j,\delta_{j+1}^j}(C) \in \mathcal{L}(D,D)$,

  and

$$\begin{aligned}\|\Theta_{U_{j+1}^j,\delta_{j+1}^j}(C)\|_{\mathcal{L}(D,D)} &\leq \|C\|_{\mathcal{L}(D,H)}\,\{\|\mathbf{E}(U_{j+1}^j)\|_{\mathcal{L}(D,D)}\,\|\mathbf{E}(\delta_{j+1}^j)\|_{\mathcal{L}(D,H)} \\ &\quad +\|U_{j+1}^j-\mathbf{E}(U_{j+1}^j)\|_{\mathcal{L}_s^2(D,D)}\,\|\delta_{j+1}^j\|_{\mathcal{L}_s^2(D,H)}\} \\ &\leq \|C\|_{\mathcal{L}(H,H)}\{\alpha_1(t_{j+1}-t_j)^{r/2+1} \\ &\quad +\alpha_0(t_{j+1}-t_j)^{(r+1)/2}(e^{\beta_D^2(t_{j+1}-t_j)}-1)^{1/2}\} \\ &\leq \|C\|_{\mathcal{L}(H,H)}(t_{j+1}-t_j)^{r/2+1}\{\alpha_1+\alpha_0\beta_D e^{\frac{1}{2}\beta_D^2(t_{j+1}-t_j)}\} .\end{aligned}$$

- $U_j \in \mathcal{L}_s^2(D,D)$, therefore for all $C \in \mathcal{L}(D,D)$

  $\Theta_{U_j,U_j}(C) \in \mathcal{L}(D,D)$,

  and

$$\|\Theta_{U_j,U_j}(C)\|_{\mathcal{L}(D,D)} \leq \|C\|_{\mathcal{L}(D,D)}\,\|U_j\|^2_{\mathcal{L}_s^2(D,D)} \leq \|C\|_{\mathcal{L}(D,D)}\, e^{\beta_D^2 t_j} .$$

It follows from this second analysis that for all $\bar{u} \in D$

$$\begin{aligned}|\mathbf{E}(\Delta_i\bar{u},\Delta_j\bar{u})_H| &\leq \{\alpha_1+\alpha_0\beta_D\, e^{\frac{1}{2}\beta_D^2(t_{j+1}-t_j)}\}\,\{\alpha_1+\alpha_0\gamma_H\, e^{\frac{1}{2}\gamma_H^2(t_{i+1}-t_i)}\} \\ &\quad (t_{i+1}-t_i)^{r/2+1}(t_{j+1}-t_j)^{r/2+1}\, e^{\gamma_H^2(T-t_{i+1})}\, e^{\frac{1}{2}(\gamma_H^2+\beta_D^2)(t_i-t_{j+1})}\, e^{\beta_D^2 t_j}\, |\bar{u}|_D^2 \\ &\leq \{\alpha_1+\alpha_0 c\, e^{\frac{1}{2}c^2k}\}^2 k^{r+2}\, e^{c^2T}\, |\bar{u}|_D^2 ,\end{aligned}$$

where $c \triangleq \max(\beta_D,\gamma_H)$.

□ *Conclusion*

Each of the square terms in (2.15) is of order $O(k^{r+1})$ and there is n such terms. On the other hand, each of the double–product terms in (2.15) is of order $O(k^{r+2})$ and there is $n(n-1)/2$ such terms. Therefore, for all $\bar{u} \in D$

$$\mathbf{E}|U_n \bar{u} - V_n \bar{u}|_H^2 \leq \alpha_0^2 T k^r \, e^{c^2 T} \, |\bar{u}|_D^2 + \{\alpha_1 + \alpha_0 c \, e^{\frac{1}{2}c^2 k}\}^2 T^2 k^r \, e^{c^2 T} \, |\bar{u}|_D^2 \, .$$

In other words

$$\|U_n - V_n\|_{\mathcal{L}_s^2(D,H)} \leq C(T) k^{r/2} \, .$$

□

The rest of this paper is devoted to the application of the approximation theorem to the time–discretization of bilinear stochastic PDE, such as Zakaï equation of nonlinear filtering.

3 Bilinear stochastic PDE

The purpose of this Section is to study the following abstract equation

$$
\begin{aligned}
du_t + Au_t\,dt &= \sum_{i=1}^{d} B_i u_t\,dY_t^i\ ,\\
u_0 &= \bar{u}\ ,
\end{aligned}
\tag{3.1}
$$

under the hypotheses listed below.

Hypotheses

- $(Y_t;\ t \geq 0)$ is a d–dimensional standard Wiener process, defined on an underlying probability space $(\Omega, \mathcal{F}, \mathrm{P})$. In particular, the σ–algebras

  $$\mathcal{Y}_t \triangleq \sigma(Y_s\ ;\ 0 \leq s \leq t)\ , \qquad \mathcal{Y}_t^s \triangleq \sigma(Y_\tau - Y_s\ ;\ s \leq \tau \leq t)\ ,$$

  satisfy, for all $s \leq t \leq u \leq v$

  $\mathcal{Y}_t^s$ and $\mathcal{Y}_v^u$ are mutually independent,

  $\mathcal{Y}_t^s \vee \mathcal{Y}_v^u \subset \mathcal{Y}_v^s$.

- Let V and H be two separable Hilbert spaces with H identified with its dual, and V densely and continuously included in H. $|\cdot|$ and $\|\cdot\|$ will denote the norm in H and V respectively, and $<\cdot,\cdot>$ the duality product between V and V'.

  Hypothesis [A]: The operator $A \in \mathcal{L}(V, V')$ is an unbounded linear operator in H. In addition, for all $u \in V$

  $$< Au, u > +\lambda |u|^2 \geq \mu \|u\|^2\ .$$

  There is no loss in generality in assuming that $\lambda = 0$, i.e.

  $$< Au, u > \geq \mu \|u\|^2\ . \tag{3.2}$$

  It follows that $-A$ generates a strongly continuous semi-group $(P_t; t \geq 0)$ of bounded linear operators in H, and

  $$\|P_t\|_{\mathcal{L}(H,H)} \leq 1\ . \tag{3.3}$$

  Moreover, A has a square root $A^{1/2}$ given by the formula [2, p.282]

  $$A^{-1/2} \triangleq \frac{1}{\pi} \int_0^{+\infty} \lambda^{-1/2} (A + \lambda I)^{-1}\, d\lambda\ . \tag{3.4}$$

  For every integer r, introduce $D^r \triangleq D(A^{r/2})$. These spaces are all separable Hilbert spaces, with norm $|\cdot|_r$. It is assumed that

  $$D(A) = D(A^*)\ , \tag{3.5}$$

where A^* denotes the adjoint of A in $\mathcal{L}(V, V')$. According to [5], this is a sufficient condition for

$$D(A^{1/2}) = D(A^{*1/2}) = V \ , \tag{3.6}$$

to hold.

- Hypothesis [B(0)]: The operators $B_i \in \mathcal{L}(H, H)$ $1 \leq i \leq d$, and by definition

$$\beta_0 \triangleq \left(\sum_{i=1}^{d} \|B_i\|_{\mathcal{L}(H,H)} \right)^{1/2} < +\infty \ . \tag{3.7}$$

In the sequel, it might be needed that these operators are more regular, e.g. satisfy for some integer r

Hypothesis [B(r)]: The operators $B_i \in \mathcal{L}(D^r, D^r)$ $1 \leq i \leq d$, in which case by definition

$$\beta_r \triangleq \left(\sum_{i=1}^{d} \|B_i\|_{\mathcal{L}(D^r,D^r)} \right)^{1/2} < +\infty \ . \tag{3.8}$$

- Finally, given any separable Hilbert space F, $M^2(0, T; F)$ will denote the subspace of those elements of $L^2(\Omega \times [0, T]; F)$ that are adapted to the filtration $(\mathcal{Y}_t \ : \ 0 \leq t \leq T)$.

3.1 Existence, uniqueness and regularity results

The following theorem [7] proves existence and uniqueness of a solution to (3.1).

Theorem 3.1 *Assume* [A], [B(0)] *and* $\bar{u} \in H$. *Then equation (3.1) has a unique solution* $u \in M^2(0, T; V)$, *which satisfies*

(i) $\quad u \in L^2(\Omega; C([0, T]; H))$,

(ii) $\quad |u_t|^2 + 2\int_0^t < Au_s, u_s > ds = |\bar{u}|^2 + 2\sum_{i=1}^{d} \int_0^t (B_i u_s, u_s)\, dY_s^i + \sum_{i=1}^{d} \int_0^t |B_i u_s|^2\, ds$.

Moreover the following estimate holds, with β_0 defined by (3.7)

$$\mathbf{E}|u_t|^2 \leq \mathbf{E}|u_s|^2 \, e^{\beta_0^2 (t-s)} \ . \tag{3.9}$$

With additional assumptions on both the initial condition $\bar{u}$ and the operators B_i $1 \leq i \leq d$, the following regularity result holds for the solution of equation (3.1).

Proposition 3.2 *Assume* [A] *and* [B(0)]. *Assume* [B(r)] *and* $\bar{u} \in D^r$ *for some integer* r. *Then the unique solution* u *of equation (3.1) satisfies*

$$u \in M^2(0,T;D^{r+1}) \cap L^2(\Omega;C([0,T];D^r)) .$$

PROOF. The proof presented here is adapted from [1]. In fact, it will be proved by induction with respect to r, that

If $[\mathrm{B}(r)]$ *holds and* $\bar{u} \in D^r$,

then $v \triangleq A^{r/2}u$ *is the unique solution in* $M^2(0,T;V)$ *of*

$$\begin{aligned} dv_t + Av_t\,dt &= \sum_{i=1}^{d} A^{r/2}B_i u_t\,dY_t^i \\ v_0 &= A^{r/2}\bar{u} \end{aligned} \tag{3.10}$$

and therefore satisfies

(i) $v \in L^2(\Omega;C([0,T];H))$,

(ii)
$$\begin{aligned} |v_t|^2 + 2\int_0^t < Av_s, v_s > ds = |A^{r/2}\bar{u}|^2 &+ 2\sum_{i=1}^{d}\int_0^t (A^{r/2}B_i u_s, v_s)\,dY_s^i \\ &+ \sum_{i=1}^{d}\int_0^t |A^{r/2}B_i u_s|^2\,ds . \end{aligned}$$

The assertion holds for $r = 0$, by Theorem 3.1.

Suppose now that it holds for a given integer r, and asume that $[\mathrm{B}(r+1)]$ holds and $\overline{u} \in D^{r+1}$. A fortiori $\overline{u} \in D^r$ and also $[\mathrm{B}(r)]$ holds by interpolation. By induction hypothesis, it follows that $u \in M^2(0,T;D^{r+1})$.

Next, define $J_\varepsilon \triangleq (I + \varepsilon A^{1/2})^{-1}$ and $J_\varepsilon^* \triangleq (I + \varepsilon A^{*1/2})^{-1}$. Property (3.6) implies that both J_ε and J_ε^* belong to $\mathcal{L}(H,V)$. Therefore, J_ε itself belongs to each of the three spaces $\mathcal{L}(V',V')$, $\mathcal{L}(H,H)$ and $\mathcal{L}(V,V)$, and so does $A^{1/2}J_\varepsilon = (I - J_\varepsilon)/\varepsilon$. Now, for all $v \in V$

$$A^{1/2}J_\varepsilon Av = AA^{1/2}J_\varepsilon v \tag{3.11}$$

since this equality obviously holds for any $v \in D(A)$ by (3.4), $D(A)$ is dense in V, and the operators on both sides of (3.11) belong to $\mathcal{L}(V,V')$.

Define next $v^\varepsilon \triangleq A^{1/2}J_\varepsilon v$ where v is the unique solution of (3.10). First $v^\varepsilon = J_\varepsilon A^{(r+1)/2}u$ and since $u \in M^2(0,T;D^{r+1})$ it follows that

$$v^\varepsilon \longrightarrow A^{(r+1)/2}u \ , \ \text{ in } M^2(0,T;H) . \tag{3.12}$$

On the other hand, since $A^{1/2}J_\varepsilon \in \mathcal{L}(V',V')$, it follows that v^ε satisfies

$$\begin{aligned} dv_t^\varepsilon + A^{1/2}J_\varepsilon Av_t\,dt &= \sum_{i=1}^{d} J_\varepsilon A^{(r+1)/2}B_i u_t\,dY_t^i \\ v_0^\varepsilon &= J_\varepsilon A^{(r+1)/2}\bar{u} \end{aligned} \tag{3.13}$$

Using (3.11) gives

$$dv_t^\varepsilon + Av_t^\varepsilon\, dt = \sum_{i=1}^{d} J_\varepsilon \phi_t^i\, dY_t^i \ ,$$

with $\phi_t^i \triangleq A^{(r+1)/2} B_i u_t$.

Since $B_i \in \mathcal{L}(D^{r+1}, D^{r+1})$ for $1 \leq i \leq d$, and $u \in M^2(0,T; D^{r+1})$, it follows that $\phi^i \in M^2(0,T;H)$ and therefore $J_\varepsilon \phi^i \longrightarrow \phi^i$ in $M^2(0,T;H)$. Also $v_0^\varepsilon \longrightarrow A^{(r+1)/2}\bar{u}$ in H. It is now easy to prove, along the lines of the proof of Theorem 1.1 in [7], that any subsequence of $(v^\varepsilon : \varepsilon > 0\}$ is a Cauchy sequence in both $M^2(0,T;V)$ and $L^2(\Omega; C([0,T];H))$. But in view of (3.12) the limit has to be $A^{(r+1)/2}u$. □

Moreover the following estimate holds, with β_r defined by (3.8)

$$\mathbf{E}|u_t|_r^2 \leq \mathbf{E}|u_s|_r^2\, e^{\beta_r^2(t-s)} \ . \tag{3.14}$$

3.2 Associated stochastic semi–group

Theorem 3.1 allows to define a two–parameter family $(U_t^s : 0 \leq s \leq t)$ of linear operators from H into $L^2(\Omega;H)$ in the following way: for all $\bar{u} \in H$, $U_t^s\bar{u}$ is the value at time t of the unique solution of equation (3.1) starting from the initial condition $\bar{u}$ at time s.

Obviously, the following properties hold

- the linear operator U_t^s is continuous from H into $L^2(\Omega;H)$, by estimate (3.9),
- for all $\bar{u} \in H$, the random variable $U_t^s\bar{u}$ is $\mathcal{Y}_t^s$–measurable,
- for all $s \leq t \leq u$, $U_u^s = U_u^t\, U_t^s$,
- for all $\bar{u} \in H$, $\lim_{k \to 0^+} \mathbf{E}|U_{t+k}^t\bar{u} - \bar{u}|_D^2 = 0$, by property *(i)* of Theorem 3.1.

Therefore

Proposition 3.3 *The two-parameter family* $(U_t^s : 0 \leq s \leq t)$ *is a strongly continuous stochastic semi–group in* H*, with first two moments*

$$\mathbf{E}(U_t^s) \ = \ P_{t-s} \tag{3.15}$$

$$\|U_t^s\|_{\mathcal{L}_2^2(H,H)} \ \leq \ e^{\frac{1}{2}\beta_0^2(t-s)} \tag{3.16}$$

where $(P_t : t \geq 0)$ *is the (deterministic) semi-group generated by* $-A$*, and* β_0 *is defined by (3.7).*

Again, with additional assumptions on both the initial condition $\overline{u}$ and the operators B_i $1 \leq i \leq d$, more precise results are available

Proposition 3.4 *If (3.8) holds for some integer r, then the two-parameter family $(U_t^s : 0 \leq s \leq t)$ is a strongly continuous stochastic semi-group in D^r. Moreover*

$$\|U_t^s\|_{\mathcal{L}_s^2(D^r,D^r)} \leq e^{\frac{1}{2}\beta_r^2(t-s)} \tag{3.17}$$

where β_r is defined by (3.8).

Remark. Estimates (3.16) and (3.16) are mere restatements of estimates (3.9) and (3.14) respectively.

From the perturbation representation

$$U_t^s = P_{t-s} + \sum_{i=1}^{d} \int_s^t P_{t-r} B_i U_r^s \, dY_r^i \tag{3.18}$$

the following estimates are easily derived

$$\|U_t^s - \mathbf{E}(U_t^s)\|_{\mathcal{L}_s^2(H,H)} \leq (e^{\beta_0^2(t-s)} - 1)^{1/2} \; ,$$

$$(\text{resp. } \; \|U_t^s - \mathbf{E}(U_t^s)\|_{\mathcal{L}_s^2(D^r,D^r)} \leq (e^{\beta_r^2(t-s)} - 1)^{1/2} \;) \; ,$$

using (3.3) and (3.7), (3.16) (resp. (3.8), (3.17)).

4 Some product formulas

The purpose of this Section is to propose and study time–discretization schemes for equation (3.1).

$$du_t + Au_t\,dt = \sum_{i=1}^{d} B_i u_t\,dY_t^i\,,$$

$$u_0 = \bar{u}\,.$$

Remark first that two operators are involved in this equation

- the unbounded operator A, with a deterministic contribution,
- the bounded operators $(B_1, B_2, \cdots, B_d)$, with a stochastic contribution.

If only A were present (i.e. $B_1 = B_2 = \cdots = B_d = 0$), then the associated semi–group would be $(P_t\,:\,t \geq 0)$ i.e. the semi–group generated by $-A$. On the other hand, if only $(B_1, B_2, \cdots, B_d)$ were present (i.e. $A = 0$), then the associated two–parameter semi–group would be

$$\Psi_t^s \triangleq \exp\left[\sum_{i=1}^{d} B_i\,(Y_t^i - Y_s^i) - \frac{1}{2}\sum_{i=1}^{d} B_i^2\,(t-s)\right]\,. \tag{4.1}$$

Therefore, it seems natural to consider the following numerical scheme. Let $\pi\,:\,0 = t_0 < t_1 < \cdots < t_i < \cdots < t_n = T$ be a given partition of the interval $[0, T]$, with mesh–size k. The proposed approximation $\bar{u}_n$ of u_{t_n} – the value at time t_n of the solution to equation (3.1) – is given by the following recursion

$$\begin{aligned} \bar{u}_{n+1} &= \left[P_{t_{n+1}-t_n}\,\Psi_{t_{n+1}}^{t_n}\right]\,\bar{u}_n \\ \bar{u}_0 &= \bar{u} \end{aligned} \tag{4.2}$$

This is nothing but a Trotter–like product formula. A next step would be to approximate the deterministic operator $P_{t_{n+1}-t_n}$ by a simple and computable one, involving only the generator $-A$. This is a rather standard part, for which some possible answers are

- implicit Euler scheme,
- Crank–Nicholson scheme.

In any case, some desirable properties of any discretization scheme for equation (3.1) should include

- decoupling of the deterministic part and the (generally straightforward to deal with) stochastic part,

- (within the context of nonlinear filtering) availability of a probabilistic interpretation.

The latter issue will be discussed in detail in another paper. In concrete situations, the former property will make the analysis of complete discretization schemes (i.e. including an additional discretisation with respect to a "space variable") quite easy.

The analysis made in the next subsection will show that the rate of convergence of the scheme (4.2) is of order $O(k)$.

Obviously, the proof of this result will rely on Theorem 2.10. Remark just that the discrete stochastic semi-group defined by the family $(U_{t_{i+1}}^{t_i} : i = 0, 1, \cdots)$ already satisfies some of the hypotheses of Theorem 2.10. To prove the needed estimates on the one-step error, one possible approach would be to use a stochastic Taylor formula like in [6,9,10]. Such a stochastic Taylor formula would indeed not be difficult to prove in the context of bilinear stochastic PDE, along the lines of Theorem 2.1 in Newton [6, pp.25-26]. However, it is a general situation in the infinite-dimensional setting, that Taylor formulas (either deterministic or stochastic) *do not* provide suitable schemes. In particular, they are explicit schemes, generally unstable because of the unboundedness of the deterministic operator A.

4.1 Rate of convergence of order $O(k)$

Generally speaking, if $B_i \in \mathcal{L}(D^r, D^r)$ $(i = 1, 2, \cdots, d)$ for some integer r, then $(\Psi_t^s : 0 \leq s \leq t)$ is a strongly continuous stochastic semi-group in D^r. Moreover

$$\mathbf{E}(\Psi_t^s) = I , \tag{4.3}$$

$$\|\Psi_t^s\|_{\mathcal{L}_2^2(D^r, D^r)} \leq e^{\frac{1}{2}\beta_r^2(t-s)} . \tag{4.4}$$

Theorem 4.1 *Let u denote the unique solution of equation (3.1). If $B_i \in \mathcal{L}(D^2, D^2)$ $1 \leq i \leq d$ and $\bar{u} \in D^2$, then the rate of convergence of the discretization scheme defined by (4.2) is of order $O(k)$, i.e. for all $\bar{u} \in D^2$*

$$\left(\mathbf{E}|u(t_n) - \bar{u}_n|^2\right)^{1/2} \leq C^{st} \cdot k|\bar{u}|_2 .$$

PROOF. Consider the two following discrete strong stochastic semi-groups defined by

$$U_{i+1}^i \triangleq U_{t_{i+1}}^{t_i} , \quad V_{i+1}^i \triangleq V_{t_{i+1}}^{t_i} ,$$

where

$$V_t^s \triangleq P_{t-s}\, \Psi_t^s .$$

It follows from (4.3) and (4.4) respectively that

$$\mathbf{E}(V_t^s) = P_{t-s} , \tag{4.5}$$

$$\|V_t^s\|_{\mathcal{L}_2^2(D^2, D^2)} \leq e^{\frac{1}{2}\beta_2^2(t-s)} . \tag{4.6}$$

Now (3.15) and (4.5) imply

$$\mathbf{E}(\delta_t^s) = 0 \; .$$

According to Theorem 2.10, it will be enough to get some estimates on the one–step error $\delta_{i+1}^i \triangleq U_{i+1}^i - V_{i+1}^i$.

By Itô formula

$$\begin{aligned} V_t^s &= P_{t-s} \left(I + \sum_{i=1}^d \int_s^t B_i \, \Psi_\tau^s \, dY_\tau^i \right) \\ &= P_{t-s} + \sum_{i=1}^d \int_s^t P_{t-\tau} \, P_{\tau-s} \, B_i \, \Psi_\tau^s \, dY_\tau^i \\ &= P_{t-s} + \sum_{i=1}^d \int_s^t P_{t-\tau} \, B_i \, V_\tau^s \, dY_\tau^i + \sum_{i=1}^d \int_s^t P_{t-\tau} \, (P_{\tau-s} \, B_i - B_i \, P_{\tau-s}) \, \Psi_\tau^s \, dY_\tau^i \; . \end{aligned}$$

By difference with (3.18)

$$\delta_t^s \triangleq U_t^s - V_t^s = \sum_{i=1}^d \int_s^t P_{t-\tau} \, B_i \, \delta_\tau^s \, dY_\tau^i + \sum_{i=1}^d \int_s^t P_{t-\tau} \, (P_{\tau-s} \, B_i - B_i \, P_{\tau-s}) \, \Psi_\tau^s \, dY_\tau^i \; .$$

Therefore, for all $x \in H$

$$\mathbf{E}|\delta_t^s x|^2 \le 2 \sum_{i=1}^d \mathbf{E} \int_s^t |P_{t-\tau} \, B_i \, \delta_\tau^s x|^2 \, d\tau + 2 \sum_{i=1}^d \mathbf{E} \int_s^t |P_{t-\tau} \, (P_{\tau-s} \, B_i - B_i \, P_{\tau-s}) \, \Psi_\tau^s x|^2 \, d\tau \; . \quad (4.7)$$

The next step is to get some estimate on $(P_t \, B_i - B_i \, P_t)$. Since $B_i \in \mathcal{L}(D^2, D^2)$ $1 \le i \le d$, it follows that $(A \, B_i - B_i \, A) \in \mathcal{L}(D^2, H)$. Introducing, as usual in perturbation problems, the operator $P_{t-s} \, B_i \, P_s$, and differentiating with respect to s, gives for all $x \in D^2$

$$(P_t \, B_i - B_i \, P_t)x = \int_0^t P_{t-s} \, (A \, B_i - B_i \, A) \, P_s x \, ds \; .$$

Therefore

$$|(P_t \, B_i - B_i \, P_t)x| \le \alpha_i \, t \, |x|_2 \; ,$$

where

$$\alpha_i \triangleq \|A \, B_i - B_i \, A\|_{\mathcal{L}(D^2,H)} \; .$$

Combining (4.7), (4.4) and (4.1) gives

$$\mathbf{E}|\delta_t^s x|^2 \le 2\beta_0^2 \int_s^t \mathbf{E}|\delta_\tau^s x|^2 \, d\tau + 2\alpha^2 \int_s^t (\tau - s)^2 \, e^{\beta_2^2(\tau - s)} \, d\tau \, |x|_2^2 \; ,$$

where

$$\alpha \triangleq \left(\sum_{i=1}^d \alpha_i^2 \right)^{1/2} \; .$$

Gronwall's lemma now implies that

$$\|\delta_t^s\|_{\mathcal{L}_s^2(D^2,H)} \le C^{st} \cdot \alpha(t-s)^{3/2} \; .$$

□

It can be seen from the proof that if $(A\ B_i - B_i\ A) \in \mathcal{L}(D^1, H)$, then less regularity is required on the data. Indeed

Theorem 4.2 *Let u denote the unique solution of equation (3.1). Assume that $B_i \in \mathcal{L}(D^1, D^1)$ $1 \leq i \leq d$ and $\bar{u} \in D^1$. If in addition*

$$(A\ B_i - B_i\ A) \in \mathcal{L}(D^1, H)\ , \quad (i = 1, \cdots, d) \tag{4.8}$$

then the rate of convergence of the discretization scheme defined by (4.2) is of order $O(k)$, i.e. for all $\bar{u} \in D^1$

$$\left(\mathbf{E}|u(t_n) - \bar{u}_n|^2\right)^{1/2} \leq C^{st} \cdot k|\bar{u}|_1\ .$$

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A LIE ALGEBRAIC CRITERION FOR NON–EXISTENCE OF FINITE DIMENSIONALLY COMPUTABLE FILTERS

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1. Introduction

In this paper we shall prove a result about linear, stochastic partial differential equations and apply it to the question of exact, finite-dimensional recursive computation of optimal filters. Let $\{Y(t), 0 \leq t \leq T\}$ be an $\mathbb{R}^p$-valued Brownian motion. Throughout, we assume that Y is the canonical process on $(\Omega, \mathcal{F}, P)$, where $\Omega = \{f \in C([0,T]; \mathbb{R}^p), f(0) = 0\}$, $\mathcal{F}$ is the σ-algebra of Borel sets of Ω w. r. t. sup norm topology, and P is Wiener measure. On $\mathbb{R}^d$ define the operator

$$A = \sum_{i,j=1}^{d} a_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i} + c(x),$$

and assume that $a(x) = [a_{i,j}(x)]_{1 \leq i,j \leq d}$ is symmetric. We shall consider the solution $p(x,t)$ to the stochastic p. d. e.

$$dp(x,t) = Ap(x,t)\,dt + \sum_{i=1}^{p} h_i(x) p(x,t)\,dY^i(t) \tag{1.1}$$

$$p(\cdot, 0) = \delta_{x_0}(\cdot). \tag{1.2}$$

Sometimes we shall write $p(\cdot, t|Y)$ to emphasize the dependence of p on Y. Suppose that a set of linearly independent functions $\{\phi_1, \ldots, \phi_n\} \subset L^2(\mathbb{R}^d)$ is given, and form the random vector $\Phi_t^{(n)}(Y) = (\langle \phi_1, p(\cdot, t|Y)\rangle, \ldots, \langle \phi_n, p(\cdot, t|Y)\rangle)$ on $(\Omega, \mathcal{F}, P)$. Here $\langle \phi, \psi \rangle = \int \phi(x)\psi(x)\,dx$. Our main result, Theorem 1.1, applies the stochastic calculus of variations, or Malliavin calculus, to (1.1) in the case that A is uniformly elliptic

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and all the coefficients are analytic functions. It states Lie algebraic conditions under which the probability distribution of $\Phi_t^{(n)}$ admits a density with respect to Lebesgue measure on $\mathbb{R}^n$ for any n. This result is a refinement of a similar theorem proved in Ocone[18], in which the coefficients are assumed to be only infinitely differentiable, but the initial condition $p(\cdot,0)$ is assumed to be smooth. Introducing the analyticity condition not only allows non-smooth initial conditions, but also leads to a simpler Lie algebraic criterion that is easier to check.

To state Theorem 1.1 we introduce the following notation. Let Λ denote the Lie algebra of operators generated by $\tilde{A} = A - 1/2\sum_1^p h_i^2(x)$ and (multiplication by) $h_i(\cdot)$, $1 \le i \le p$, using the Lie bracket $[B,C] = C \circ B - B \circ C$. The elements of Λ are all partial differential operators with variable coefficients. For $x_0 \in \mathbb{R}^d$, let $\Lambda(x_0)$ denote the linear space of operators consisting of the operators of Λ with their coefficients frozen at x_0; thus, for example, $x\frac{\partial}{\partial x} \in \Lambda$ and $x_0 \neq 0$ imply $\frac{\partial}{\partial x} \in \Lambda(x_0)$. Also, given $f \in C^\infty(\mathbb{R}^d)$, $\{(Bf)(x_0),\, B \in \Lambda\} = \{(Cf)(x_0),\, C \in \Lambda(x_0)\}$. Finally, let $C_b^\omega(\mathbb{R}^d)$ denote the space of real valued, bounded functions on $\mathbb{R}^d$ which are analytic at each point of $\mathbb{R}^d$ and whose derivatives of all orders are bounded. Also, let $H^k(\mathbb{R}^d)$ be the Sobolev space of (integral) order k with norm $||f||_k^2 = \sum_{|\alpha|\le k} ||\partial^\alpha f||_{L^2}^2$.

Theorem 1.1 . *Assume that*

$$a(x) > \epsilon I \quad \text{for some } \epsilon > 0 \text{ and all } x \in \mathbb{R}^d, \tag{1.3}$$

$$a_{i,j}(\cdot),\ b_i(\cdot),\ c(\cdot),\ h_k(\cdot) \in C_b^\omega \quad \text{for } 1 \le i,j \le d,\ 1 \le k \le p, \tag{1.4}$$

$$\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}} \in \Lambda(x_0) \quad \text{for every multi-index } \alpha. \tag{1.5}$$

Then for any $t > 0$, for any n, for any linearly independent set $\{\phi_1, \ldots, \phi_n\} \subset L^2(\mathbb{R}^d)$, the probability distribution of $\Phi_t^{(n)}$ admits a density with respect to Lebesgue measure.

Section 2 of this paper gives the proof of Theorem 1.1.

Remark. Because of (1.3) and (1.4), equation (1.1)-(1.2) has a unique, adapted solution satisfying

$$E\int_0^T ||p(t)||_k^2\, dt < \infty \quad \text{for } k < -d/2 \text{ and } T > 0.$$

Moreover, $p(\cdot,t) \in C([0,T]; H^k(\mathbb{R}^d))$ a. s. for all k, and $p(\cdot,t) \in C^\infty(\mathbb{R}^d)$ for all $t > 0$ a. s. These facts are proved in Pardoux[21], see especially pp.227-228. For this

reason, $\langle \phi, p(\cdot,t)\rangle$ is well-defined for any $\phi \in \cup_{k\le 0} H^k(\mathbb{R}^d)$. The reasoning used to prove Theorem 1.1 can easily be extended to show that the theorem is still true if the condition $\{\phi_1,\ldots,\phi_n\} \subset L^2(\mathbb{R}^d)$ is replaced by $\{\phi_1,\ldots,\phi_n\} \subset H^k(\mathbb{R}^d)$ for any $k \le 0$.

We can apply Theorem 1.1 to the nonlinear filtering problem

$$dX(t) = b(X(t))\,dt + \sigma(X(t))\,dW(t), \quad X(0) = x_0 \tag{1.6}$$

$$dY(t) = h(X(t))\,dt + dB(t), \quad Y(0) = 0 \tag{1.7}$$

where W and B are, respectively, $\mathbb{R}^l$ and $\mathbb{R}^p$-valued Brownian motions, $X(t)$ evolves in $\mathbb{R}^d$, and $Y(t)$ evolves in $\mathbb{R}^p$. Let $a(x) = \sigma\sigma^T(x)$. In compliance with (1.3) and (1.4), we shall assume

$$a(x) > \epsilon I, \qquad \text{and} \quad a_{i,j},\, b_i,\, h_k \in C_b^\omega. \tag{1.8}$$

Let $p(\cdot,t|Y)$ denote the solution to

$$dp(x,t) = A_0 p(x,t)\,dt + \sum_1^p h_i(x)p(x,t)\,dY^i(t) \tag{1.9}$$

$$p(\cdot,0) = \delta_{x_0}(\cdot)$$

where $A_0u(x) = 1/2\sum \frac{\partial^2}{\partial x_i \partial x_j}(a_{i,j}(x)u(x)) - \sum \frac{\partial}{\partial x_i}(b_i(x)u(x))$. $p(\cdot,t|Y)$ is an unnormalized conditional density of $X(t)$ given the sigma algebra $\mathcal{F}_t^Y = \sigma\{Y(s),\, s \le t\}$. In (1.6)-(1.7) Y is not a Brownian motion, but the measure induced by Y on Ω is absolutely continuous with respect to Wiener measure. Hence the conclusion of Theorem 1.1 will not be affected when we apply it to (1.9).

Recently there has been interest in determining when conditional statistics, such as $\langle \phi, p(\cdot,t)\rangle$, do or do not admit finite dimensional, recursive realizations, and Theorem 1.1 has implications for this question. We shall say that the collection of statistics $\{\langle \phi_i, p(\cdot,t)\rangle,\ 1 \le i < \infty\}$ admits a finite dimensional, regular sufficient statistic α, if $\alpha : \Omega \to M$ is a measurable map into a finite dimensional, C^1–manifold, such that, for each i, there is a $\theta_i \in C^1(M;\mathbb{R})$ with

$$\langle \phi_i, p(\cdot,t)\rangle = \theta_i(\alpha) \quad a.s.$$

Let

$$\pi(\cdot,t|Y) = \frac{p(\cdot,t|Y)}{\int p(x,t|Y)\,dx}$$

denote the normalized conditional density of $X(t)$ given $\mathcal{F}_t^Y$. We shall say that

$$\{\langle \phi_i, \pi(\cdot,t)\rangle,\ 1 \le i < \infty,\ t > 0\}$$

admits a finite dimensional, regular, recursive, sufficient statistic if for each i there is a $\theta_i \in C^1(M; \mathbb{R})$ with $\langle \phi_i, \pi(\cdot,t)\rangle = \theta_i(\alpha(t))$ where

$$d\alpha(t) = f(\alpha(t))\,dt + \sum_1^p g_i(\alpha(t))\,dY^i(t) \tag{1.10}$$

for some C^1–vector fields f and g_i, $1 \le i \le p$ on M.

Corollary 1.2. *Assume (1.8) and let Λ be the Lie algebra generated by $A_0 - \sum h_i^2$ and $h_1, \dots, h_p$. If Λ satisfies (1.5), there is no countably infinite, linearly independent set $\{\phi_i, 1 \le i < \infty\} \subset L^2(\mathbb{R}^d)$ such that either $\{\langle \phi_i, p(\cdot,t)\rangle, 1 \le i < \infty\}$ admits a finite dimensional, regular sufficient statistic for any $t > 0$, or $\{\langle \phi_i, \pi(\cdot,t)\rangle, 1 \le i < \infty, t > 0\}$ admits a finite dimensional, regular, recursive sufficient statistic.*

Proof. The conclusion concerning $\{\langle \phi_i, \pi(\cdot,t)\rangle, 1 \le i < \infty, t > 0\}$ follows from that about $\{\langle \phi_i, p(\cdot,t)\rangle, 1 \le i < \infty\}$ by the identity

$$p(\cdot,t|Y) = \pi(\cdot,t|Y)\exp[\int_O^t \hat{h}_i(s)\,dY^i(s) - 1/2\int_0^t |\hat{h}(s)|^2\,ds].$$

where $\hat{h}_i(s) = \int h(x)\pi(x,t)\,dx$. To prove the result about $\{\langle \phi_i, p(\cdot,t)\rangle, 1 \le i < \infty\}$, let us assume that a finite dimensional, regular statistic exists and derive a contradiction to Theorem 1.1. Existence of such an α implies that for any n, $\Phi_t^{(n)} = (\langle \phi_1, p(\cdot,t)\rangle, \dots, \langle \phi_n, p(\cdot,t)\rangle) = (\theta_1(\alpha), \dots, \theta_n(\alpha))$. However, because the θ_i are differentiable, if $n > \dim M$, the Lebesgue outer measure in $\mathbb{R}^n$ of $\{(\theta_1(m), ..., \theta(m)) \mid m \in M\}$ is zero. Thus $\Phi_t^{(n)}$ can not admit a probability density in contradiction to Theorem 1.1.

A simple example in which (1.3)–(1.5) hold is $A = \dfrac{1}{2}\dfrac{\partial^2}{\partial x^2}$, $h(x) = \cos x$, and $x_0 \notin \{n\pi, (2n+1)\pi/2, n \in \mathbb{Z}\}$.

Recently, both Lie algebraic techniques and the Malliavin calculus have been increasingly used in nonlinear filtering theory, and we wish to compare these applications with Corollary 1.2. Brockett and Clark[4] and Mitter[16] introduced Lie algebraic and

geometric methods into filtering with the insight that, in formal analogy to realization theory in differential geometric control, existence of finite dimensional, recursive filters should impose restrictions on the structure of Λ. This inspired a lot of work into the classification of the algebras Λ associated to filtering problems and into using algebraic properties to seek finite dimensionally computable optimal filters. A nice survey of this effort and related topics may be found in Marcus[13]. Also, Chaleyat-Maurel and Michel[6] and, independently, Hijab[10] rigorously developed the original suggestions of Brockett, Clark, and Mitter. For example, in [6] Chaleyat-Maurel and Michel introduce the following notion of universal finite dimensional computability, which we describe only roughly and in modified form. $\{p(\cdot,t), t \geq 0\}$ (or $\{\pi(\cdot,t), t \geq 0\}$) is universally FDC with respect to a class of infinitely differentiable test functions S if there is a system (1.10) with C^∞–vector fields such that for every $\phi \in S$ there is a $\theta \in C^\infty(M)$ with $\theta(\alpha(t)) = \langle \phi, p(\cdot,t)\rangle$ ($\theta(\alpha(t)) = \langle \phi, \pi(\cdot,t)\rangle$). By comparing the Itô derivatives of $\theta(\alpha(t))$ and $\langle \phi, p(\cdot,t)\rangle$ at $t = 0$, one can derive a relationship between Λ and the Lie algebra of vector fields on M generated by $f, g_1, \ldots, g_p$, as long as S is large enough. For an appropriate choice of S, say all infinitely differentiable ϕ so that $\langle \phi, p(\cdot,t)\rangle$ and all its Ito derivatives make sense, it is shown in [6] that $\dim \Lambda(x_0) \leq \dim M$. By way of contrast, Corollary 1.2 draws inferences about finite dimensional computability from existence of probability densities for $\Phi_t^{(n)}$ for any n. This is apriori a much stronger property and requires the stronger condition (1.5) on $\Lambda(x_0)$. However, we are able to weaken the differentiability requirements on ϕ and in the definition of finite dimensional computability.

Other applications of Malliavin calculus to filtering may be found in the work of Michel[14], Bismut and Michel[2], Ferreyra[8], and Kusuoka and Stroock[11]. These authors study the existence and smoothness of $p(x,t)$ as a function of x. That is, they determine when the unnormalized conditional distribution, as a random measure on $\mathbb{R}^d$, admits a smooth density $p(x,t)$. In this paper, we are using Malliavin's calculus to study the measure induced on a function space by the solution of Zakai's equation. On the other hand, our work is related to that of Chaleyat-Maurel[5], who studies continuity of nonlinear filters using Malliavin calculus. She gives conditions under which conditional statistics are in the Sobolev spaces on Wiener space, but does not

analyze the Malliavin covariance matrix as we do here.

2. Proof of Theorem 1.1.

Our proof relies heavily on the analysis of [18], which we shall use without repeating proofs. For simplicity of calculation, we assume throughout that $p = 1$.

We first need to define the gradient operator D on Wiener functionals. Let $F \in L^2(\Omega, P)$. It admits an Itô-Wiener expansion

$$F = \sum_{k=0}^{\infty} f_k \odot Y^k,$$

where each $f_k \in \hat{L}^2([0,T]^k)$, which is the subspace of symmetric functions in $L^2([0,T]^k)$, and where $f_k \odot Y^k$ is the multiple Wiener integral

$$f_k \odot Y^k = \int_0^T \cdots \int_0^T f_k(t_1, \ldots, t_k)\, dY(t_1) \ldots dY(t_k).$$

Let $\mathbb{D}^{1,2}$ denote the set of $F \in L^2(\Omega, P)$ satisfying

$$\sum_1^{\infty} k(k!)\|f_k\|^2_{L^2} < \infty. \tag{2.1}$$

If $F \in \mathbb{D}^{1,2}$, we may define

$$D_s F(Y) = \sum_1^{\infty} k f_k(\ldots, s) \odot Y^{k-1}, \quad 0 \le s \le T, \tag{2.2}$$

where $f_k(\ldots, s)$ is the element of $\hat{L}^2([0,T]^{k-1})$ obtained by fixing the last variable at s. Because of (2.1), the series on the right hand side converges in $L^2(\Omega \times [0,T], P \times m)$, where m denotes Lebesgue measure on $[0,T]$, and thus $D_s F(Y)$ is well defined up to sets of $P \times m$-measure zero. In fact,

$$E[\int_0^T (D_s F)^2\, ds] = \sum_1^{\infty} k(k!)\|f_k\|^2_{L^2}.$$

Next, given $F = (F_1, \ldots, F_n) \in (\mathbb{D}^{1,2})^n$, we define the Malliavin covariance matrix of F:

$$\nabla^T F \nabla F = [\int_0^T D_s F^i D_s F^j\, ds]_{1 \le i,j \le n}. \tag{2.3}$$

Let $P \circ F^{-1}$ denote the probability distribution of F; for a Borel set $A \subset \mathbb{R}^n$, $P \circ F^{-1}(A) = P(F \in A)$. The Malliavin covariance matrix is used to study the regularity properties of $P \circ F^{-1}$. For example, Bouleau and Hirsch[3] prove the following result.

Proposition 2.1. *Suppose that $F \in (\mathbb{D}^{1,2})^n$ and*

$$\nabla^T F \nabla F > 0 \quad a.s. \tag{2.4}$$

Then $P \circ F^{-1}$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^n$.

Proposition 2.1 presents the simplest application of the Malliavin covariance matrix. The theory of the Malliavin calculus shows that moment bounds on the inverse of $\nabla^T F \nabla F$ imply smoothness properties of the density $d(P \circ F^{-1})/dx$. For an introduction to the complete theory and its applications, see Ocone[20] or Michel and Pardoux[15].

We shall use Proposition 2.1 with $F = \Phi_t^{(n)}$ to prove Theorem 1.1. Notice that $\Phi_t^{(n)}$ is adapted to $\sigma\{Y(s),\, s \leq t\}$. Therefore, we may replace T by t in (2.3) in discussing $\Phi_t^{(n)}$, and so for the rest of the argument we assume $T = t$.

Before continuing, we note that it suffices to consider operators A in (1.1) of the form that appear in Zakai's equation (1.9) modulo a potential term:

$$Au(x) = \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}(\sigma\sigma^T(x)_{i,j} u(x)) - \sum_i \frac{\partial}{\partial x_i}(b_i(x)u(x)) - c(x)u(x) \tag{2.5}$$

where $\sigma(x) : \mathbb{R}^d \to \mathbb{R}^{d \times d}$. This is possible because for any $a(x) = [a_{i,j}(x)]$ satisfying (1.3) and (1.4) there is a $\sigma(x) \in C_b^\omega$ satisfying $a(x) = \sigma\sigma^T(x)$. For example, following Friedman[9], pp. 128–129, we may take $\sigma(x) = (1/2\pi)\int_\Gamma \sqrt{z}(a(x) - zI)^{-1}\, dz$, where Γ is a simple closed curve in $\Re\, z > 0$ containing all the eigenvalues of $a(x)$ for all $x \in \mathbb{R}^d$. Thus, by suitably choosing b and c we can transform any operator of the form in (1.1) to the form in (2.5). The advantage of (2.5) is that $A + c$ is the forward generator of the diffusion associated to

$$dX(t) = b(X(t))\, dt + \sigma(X(t))\, dW(t), \tag{2.6}$$

and we can then represent the solution to (1.1) with the Kallianpur-Striebel formula from nonlinear filtering. Suppose that W, and hence X are defined on a second probability space $(\Omega', \mathcal{F}', Q)$. Extend W, X, and the canonical process Y on (Ω, P) to the product probability space $(\Omega \times \Omega', \mathcal{F} \times \mathcal{F}', P \times Q)$ by $W(\omega, \omega')(t) = W(\omega')(t)$,

$Y(\omega,\omega')(t) = Y(\omega)(t)$, etc. Let E_Q denote expectation with respect to Q on Ω', let $X_{x_0}(t)$ be the solution of (2.6) with $X_{x_0}(0) = x_0$ and set

$$\begin{aligned} L(Y,t) &= \exp[\int_0^t h(X_{x_0}(s))\,dY(s) - 1/2\int_0^t h(X_{x_0}(s))^2\,ds] \\ &= \exp[h(X_{x_0}(t))Y(t) - \int_0^t Y(s)h'(X_{x_0}(s))\,dX_{x_0}(s) \\ &\quad - 1/2\int_0^t [h(X_{x_0}(s))^2 + Y(s)\mathrm{tr}(a(X_{x_0}(s))h''(X_{x_0}(s)))]\,ds \end{aligned}$$

Let $p(\cdot,t|Y)$ solve (1.1)–(1.2) with A given by (2.5). Then the Kallianpur-Striebel formula, modified by the potential c gives

$$\langle\phi, p(\cdot,t|Y)\rangle = E_Q\big[\phi(X_{x_0}(t))e^{-\int_0^t c(X_{x_0}(s))\,ds}L(Y,t)\big] \quad \text{for } P \text{ a. e. } Y. \tag{2.7}$$

(2.7) is proved in Pardoux[21] with $c = 0$, and the method extends easily to non-zero c. The right hand side of (2.7) is well defined *for every* $Y \in \Omega$, and we always use this particular version of $\langle\phi, p(\cdot,t|Y)\rangle$.

We are now in a position to calculate the gradient of $\langle\phi, p(\cdot,t|Y)\rangle$ using some nonlinear filtering theory.

Lemma 2.2. *Under the assumptions (1.4), if* $\phi \in L^2(\mathbb{R}^d)$, $\langle\phi, p(\cdot,t|Y)\rangle \in \mathbb{D}^{1,2}$ *and* $D_u\langle\phi, p(\cdot,t|Y)\rangle = 1_{[0,t]}(u)E_Q[\phi(X_{x_0}(t))h(X_{x_0}(u))\exp\{-\int_0^t c(X_{x_0}(s))\,ds\}L(Y,t)]$. *It follows that*

$$\int_0^t [D_u\langle\phi, p(\cdot,t|Y)\rangle]^2\,du = \int_0^t E_Q[\phi(X_{x_0}(t))h(X_{x_0}(u))e^{-\int_0^t c(X_{x_0}(s))\,ds}L(Y,t)]^2\,du. \tag{2.8}$$

Proof. By Theorem 3.1 in [17] $\langle\phi, p(\cdot,t|Y)\rangle = \sum_0^\infty f_k \odot Y^k$, where

$$f_k(t_1,\ldots,t_k) = \frac{1}{k!}E_Q\big[\phi(X_{x_0}(t))e^{-\int_0^t c(X_{x_0}(s))\,ds}h(X_{x_0}(t_1))\ldots h(X_{x_0}(t_k))\big].$$

We shall show that $\langle\phi, p(\cdot,t|Y)\rangle \in \mathbb{D}^{1,2}$ by verifying (2.1). If $\phi \in L^2(\mathbb{R}^d)$, Cauchy-Schwarz implies $E_Q|\phi(X_{x_0}(t))| \le ||\phi||_{L^2}||q(t)||_{L^2}$ where $q(x,t)$ is the density of $X_{x_0}(t)$. However, $q(x,t)$ solves

$$\frac{\partial q}{\partial t} = 1/2\sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}(a_{i,j}(x)q) - \sum_i\frac{\partial}{\partial x_i}(b_i(x)q), \quad q(\cdot,0) = \delta_{x_0}(\cdot),$$

and, by the analysis of Pardoux[21], pp 227–228, $||q(t)||_{L^2} < \infty$ for $t > 0$. Thus, if $M = \sup_x |c(x)|$ and $K = \sup_x |h(x)|$,

$$||f_k||_{L^2}^2 \leq e^{Mt}\frac{K^k t^k}{(k!)^2}||\phi||_{L^2}^2 ||q(t)||_{L^2}^2.$$

It follows easily that $\langle \phi, p(\cdot, t|Y)\rangle \in \mathbb{D}^{1,2}$.

Next, by (2.2), $D_u \langle \phi, p(\cdot, t|Y)\rangle = \sum g_k(\ldots, u) \odot Y^{k-1}$ for $u \leq t$, where

$$g_k(t_1, \ldots, t_{k-1}, u) = E_Q\left[\phi(X_{x_0}(t))e^{-\int_0^t c(X_{x_0}(s))\,ds} h(X_{x_0}(t_1))\ldots h(X_{x_0}(t_k))h(X_{x_0}(u))\right].$$

By again applying Theorem 3.1 in [17], this time in reverse, we use these expressions for g_k to get the result of the Lemma.

We shall next represent the integrand of (2.8) in terms of an inner product between p and the solution of a backward stochastic p. d. e. adjoint to (1.1). This will then put us exactly in the situation of [18], and we will introduce a lemma from [18] to complete the proof. Let

$$v(x, r|Y, \phi) =$$
$$E_Q\left[\phi(X_{x_0}(t))e^{-\int_r^t c(X_{x_0}(s))\,ds + \int_r^t h(X_{x_0}(s))\,dY(s) - 1/2\int_r^t h^2(X_{x_0}(s))\,ds} \mid X_{x_0}(r) = x\right].$$

Following the analysis of Pardoux[21], Chapter III,

$$v(x, r|Y, \phi) = e^{-h(x)Y(r)}u(x, r|Y, \phi) \tag{2.9}$$

where

$$\begin{aligned} \frac{\partial}{\partial r}u(x, r|Y, \phi) &= -(A_r^Y)^* u(x, r|Y, \phi) \quad r < t \\ u(x, t) &= \phi(x)e^{h(x)Y(t)} \end{aligned} \tag{2.10}$$

and $(A_r^Y)^*$ is the formal adjoint of $A_r^Y = e^{-h(x)Y(r)}[A - 1/2h^2]e^{h(x)Y(r)}$. ((2.9)–(2.10) is the robust form of what appears in Pardoux[21], except that in [20] there is no potential c.) *Now from the Markov property of* X_{x_0}

$$E_Q\left[\phi(X_{x_0}(t))e^{-\int_0^t c(X_{x_0}(s))\,ds}h(X_{x_0}(r))L(Y, t)\right] = \langle v(\cdot, r|Y, \phi), h(\cdot)p(\cdot, r|Y)\rangle.$$

Thus by (2.8), $\nabla^T\langle \phi, p(\cdot, t|Y)\rangle \nabla\langle \phi, p(\cdot, t|Y)\rangle = \int_0^t \langle v(\cdot, r|Y, \phi), h(\cdot)p(\cdot, r|Y)\rangle^2\,dr$. This is precisely the type of formula encountered in the analysis in Ocone[18] which was applied to (1.1) without the added assumption of analyticity. The analysis leading to Lemma 4.18 in [18] may be repeated with only very minor modification to obtain,

Lemma 2.3. *There is a set $\mathcal{N} \subset \Omega$ with $P(\mathcal{N}) = 0$, such that for every $Y \in \mathcal{N}^c$ and every non-zero $\phi \in L^2(\mathbb{R}^d)$,*

$$\int_0^t \langle v(\cdot, r|Y, \phi), h(\cdot)p(\cdot, r|Y)\rangle^2 \, dr = 0$$

implies $\langle v(\cdot, r|Y, \phi), Cp(\cdot, r|Y)\rangle = 0$ for $0 < r \le t$ for every $C \in \Lambda$.

Remark. The proof of Lemma 2.2 involves repeatedly differentiating the integrand with respect to r. Care must be excercised since $v(\cdot, r|Y, \phi)$ is adapted to the future and $p(\cdot, t|Y)$ to the past of Y.

Proof of Theorem 1.1. By Proposition 2.1 it suffices to show that $\nabla^T \Phi_t^{(n)} \nabla \Phi_t^{(n)} > 0$ a. s., where $\Phi_t^{(n)} = \{\langle \phi_1, p(\cdot, t)\rangle, \ldots, \langle \phi_n, p(\cdot, t)\rangle\}$. Equivalently, we must show that

$$\xi^T \nabla^T \Phi_t^{(n)} \nabla \Phi_t^{(n)} \xi > 0, \quad \text{for all non-zero } \xi \in \mathbb{R}^n, \text{ a. s.}$$

However, since $\xi^T \nabla^T \Phi_t^{(n)} \nabla \Phi_t^{(n)} \xi = \nabla^T \langle \sum_1^n \xi_i \phi_i, p(\cdot, t)\rangle \nabla \langle \sum_1^n \xi_i \phi_i, p(\cdot, t)\rangle$, it is enough to show

$$\nabla^T \langle \phi, p(\cdot, t|Y)\rangle \nabla \langle \phi, p(\cdot, t|Y)\rangle > 0 \quad \forall\, \phi \in L^2(\mathbb{R}^d),\ \phi \neq 0 \text{ and } \forall\, Y \in \mathcal{N}^c, \tag{2.11}$$

where $\mathcal{N}$ is the set found in Lemma 2.2, because $P(\mathcal{N}^c) = 0$. Suppose to the contrary that for some $\phi \neq 0$ and some $Y \in \mathcal{N}^c$, $\nabla^T \langle \phi, p(\cdot, t|Y)\rangle \nabla \langle \phi, p(\cdot, t|Y)\rangle = 0$. Then by Lemmas 2.2 and 2.3, $\langle v(\cdot, r|Y, \phi), Cp(\cdot, r|Y)\rangle = 0$, $0 < r \le t$, for all $C \in \Lambda$. Recall that $p(\cdot, r|Y) \in H^k(\mathbb{R}^d)$ for all k. Similarly, by applying the analysis of Pardoux[21], pp. 227–228, to the equation (2.10) for u, one finds that $v(\cdot, r|Y, \phi) \in H^k(\mathbb{R}^d)$ for all k. Therefore, we may integrate by parts to find that $\langle C^* v(\cdot, r|Y, \phi), p(\cdot, r|Y)\rangle$, $0 < r \le t$, for all $C \in \Lambda$. Taking $r \downarrow 0$ and using (1.2), $Cv(x_0, o|Y, \phi) = 0$ for every $C \in \Lambda_{x_0}^*$, where $\Lambda_{x_0}^*$ is the space of operators formed by freezing the coefficients of the operators in $\Lambda^* = \{C^*, C \in \Lambda\}$ at x_0. It is clear that if Λ has property (1.5), then so does Λ^*. Hence

$$\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}} v(x, 0|Y, \phi) \bigg|_{x=x_0} = 0 \quad \text{for all multi-indices } \alpha. \tag{2.12}$$

By applying Theorem 6.2, p. 221, in Eidel'man[7] to (2.10), one finds that $v(\cdot, 0|Y, \phi) \in C_b^\omega$. Hence (2.12) implies that $v(\cdot, 0|Y, \phi) \equiv 0$ and hence $u(\cdot, 0|Y, \phi) \equiv 0$. But u satisfies the backward parabolic p. d. e. (2.10), and $(A_r^Y)^*$ is uniformly elliptic because of (1.3).

Theorem II.1 of Bardos and Tatar[1] on backward uniqueness of evolution equations therefore applies and shows that $u(x,t|Y,\phi) = \phi(x)e^{h(x)Y(t)} \equiv 0$ also. This contradicts the initial assumption that $\phi \neq 0$, and completes the proof.

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